ACADEMY OF SCIENCES OF UKRAINE
INSTITUTE FOR THEORETICAL PHYSICS

D. S. SOURLAS and G. T. TSAGAS

MATHEMATICAL FOUNDATIONS of the LIE-SANTILLI THEORY

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TABLE OF CONTENTS

PREFACE

CHAPTER I : ISOALGEBRAIC STRUCTURES

I.1 : Statement of the problem	1
1.2 : Isogroup	6
1.3 : Isoring	I0
1.4 : Isofields	12
I.5 : Isovectorspaces and Isotransformations	17
I.6 : Isomodule	26
I.7 : Isospaces	26
1.8 : Isoalgebras	38

CHAPTER II LIE-SANTILLI ALGEBRAS

2.I : Statement of the problem	59
2.2 : Isotopic generalization of universal enveloping associative algebras.	75
2.3 : Isotopic generalization of Lie's first, second, and third theorems	94

CHAPTER III ISOMANIFOLDS

3.I : Real Cartecian manifold	I13
3.2 : Real Cartecian isomanifold	II7
3.3 Isomanifold	123

CHAPTER IV ISOTENSOR FIELDS ON ISOMANIFOLDS

4.1 : Isofunction	133
4.2 : Germs of differentiable isofunctions	137
4.3 : Isovector fields on an isomanifold	I4I

4.4 : Isotangent space at a point of an isomanifold 152
4.5 : Differentiable isoexterior forms of the first order 157
4.6 : Isotensor fields 162
4.7 : Subalgebras of the isotensor algebra associated to an isomanifold 176
4.8 : Mappings on the isotensor algebra D(M) 177

CHAPTER V ISOEXTERIOR ALGEBRA
5.1 : Construction of the isoexterior algebra associated to an isomanifold 182
5.2 : Local expression of an isoexterior p-form 188
5.3 : Operator on an isoalgebra 192
5.4 : Connection between Isosymplectic Geometry and Lie-isotopic Algebras 195

CHAPTER VI ISOMAPPING BETWEEN ISOMANIFOLDS
6.1 : Basic properties of isomappings 202
6.2 : Connection between isotensor fields and isomapping 213
6.3 : Subisomanifolds 217
6.4 : Isodistribution on an isomanifold 223

REFERENCES 228

CHAPTER I

ISOALGEBRAIC STRUCTURES

1.1 : STATEMENT OF THE PROBLEM

As well known, the discovery of the mathematician Sophus Lie about one century ago, today called **Lie's theory**, has a wide range of applications in various branches of contemporary physics and mathematics.

Lie groups found their way into physics even before the development of the quantum theory through their finite and infinite dimentional matrix representations. They were useful for the description of pseudo-Riemannian, (locally) homogeneous symmetric spaces, being used in particular in geometric theories of gravitations. But Lie groups were virtually forced into physics by the development of the modern quantum theory in 1925-1926. In this theory, physical observables appear through their hermitian matrix representatives, whereas processes producing transformations are described by their unitary or antiunitary matrix representations. Operators that close under commutation belong to a finite dimensional Lie algebra ; transformation processes described by a finite number of continuous parameters belong to a Lie group.

The importance of Lie algebras and Lie groups (Jacobson 1962a, Gilmore 1974a, Barut 1977a),for mathematics and for physics has also become increasingly evident in recent years. In fact, Lie's theory remains a powerful tool for studing differential equations, special functions, pertubation theory, differential geometry and other fields.

Lie's theory often enters into physics either through the presence of exact kinematical symmetries or through the use of idealized dynamical models having symmetry greater than that present in the real world. These exact kinematical symmetries include rotational, translational and Galilean or Lorentz invariance, as well as symetries arising from the use of canonical formalism in both classical and quantum mechanics. Lie's theory finds applications not only in elementary particle physics and nuclear physics, but also in such diverse fields as continuum mechanics, solid state physics, cosmology and control theory.

A generalization of Lie's theory was discovered in 1978 by the theoretical physicist Ruggero Maria Santilli (1978a) under the original name of **Lie-isotopic theory** and subsequantly called **Lie-Santilli theory**, (see Aringazin, Jannussis et al (1991), Kadeisvili (1992), and quoted references). The word "isotopic" comes from the greek words "**ισος τοπος**", which mean "the same place", that is the same configuration. The primary aim of the new theory is to generalized the conventional Lie theory in such a way to be effective for the treatment of nonlocal, (integral) and nonlagrangian , (nonhamiltonian) physical problems, where the conventional theory fails to be applicable, while preserving

the original Lie axioms.

The central point is the generalization of the conventional unit of every algebraic, analytic and geometric structure (such as field, metric space, vector space, algebra, etc) into a new unit $\hat{I}$ called **isotopic unit** or **isounit**, under the condition of preserving the original topological properties of I, in order to qualify as an isotopy. The resulting new mathematical structures include the old ones as special cases. The replacement of the old unit I by the isounit $\hat{I}$ is called isotopic **lifting**. The methods for the construction of the generalized formulations are known under the generic name of **isotopies**.

The main idea of the isotopies is that of identifying the ultimate geometric properties and axioms of the theory considered, and realizing them in their most general possible nonlinear, nonlocal and noncanonical way. This generally can be done via an infinite numbers of possible isotopies.

The conventional Lie theory has been developed in both mathematical and physical literatures with respect to the simplest conceivable unit, say I=diag(1,1,...1), and the simplest conceivable product AB-BA, where AB is the trivial associative product, say, of vector fields or matrices.

Prof. Santilli's generalized formulation of Lie's theory is constructed with respect to the most general possible unit $\hat{I}$, in which case the Lie product assumes less trivial forms, such as A*B-B*A where A*B is still associa-

tive but of the more general type $A{*}B=ATB$, where T is fixed, nonsingular and such that $\hat{I}=T^{-1}$. In the most general case the isounit $\hat{I}$ can possess a nonlinear and nonlocal dependence on time t, coordinates r, their derivatives of arbitrary order $\dot{r},\ddot{r}$, as well as any other needed quantity

$$\hat{I}=\hat{I}(t,r,\dot{r},\ddot{r}\ldots) \qquad (I.I.I)$$

In any case the isounit $\hat{I}$ must verify certain properties. As a first example, the primary properties of the conventional unit I are those of being nowhere singular, real valued and symmetric. The lifting $I \longrightarrow \hat{I}$ must then be done in such a way that the isounit $\hat{I}$ is nowhere null, to be everywhere invertible in the considered region of the local variables, and hermitean.

As we shall see, the assumption of $\hat{I}$ as the generalized unit of the theory has nontrivial mathematical implications, in as much as it implies the generalization of each and every notion used in contemporary mathematics, such as : field, metric space, Lie algebras, symplectic geometry, affine geometry, Riemannian geometry, etc. A step-by-step generalization of Lie's theory is then consequential.

In this way Santilli generalized (I978a) the enveloping associative algebras, Lie's first, second and third theorems and the conventional notion of Lie group into forms compatible with the most general possible unit $\hat{I}$. Additional subsequent studies on the isotopies of Lie's theory can be found in Santilli, (I978b; I979; I982a; I979c,I982c; I983a;

1983e; 1985a,b; 1988a,b,c,d; 1989a,b,c,d; 1983b; 1983c; 1985a,b; 1988a,b,c,d; 1989a,b,c,d; 1991a,be,f and the recent monographs 1991c,d).

Under the condition that the old unit I is contained as a particular case of the generalized unit $\hat{I}$, the Lie-isotopic theory becomes a covering of the conventional one, in the sense of being formulated on stucturaly more general foundations, while admitting the conventional formulation as a particular trivial case.

Physically, the isounit (1.1.1) has equally far reaching implications, in as much as it requires a necessary generalization of conventional space-time symmetries and, consequently, of contemporary relativities,(Santilli 1991c,d).

The basic notions of isostructures and their applications are contained in the chapters I and II. In the third chapter the isomanifolds are studied. The isotensor algebra over an isomanifold is contained in the fourth chapter. The fifth chapter includes the isoexterior algebra over an isomanifolds. The isomapping between two isomanifolds is included in the sixth paragraph. The isogroups and their associated Lie-isoalgebras are contained in the seventh chapter. The eighth paragraph include the isoconections on an isomanifold. The last chapter contains Riemannian isometric on an isomanifold.

1.2 ISOGROUPS

DEFINITION I.2.1 : Let A be a set with elements $\alpha,\beta,\gamma,\ldots$. An algebraic composition law on A is a rule, which assigns to any ordered pair (α,β) an element c of A. Thus, a composition law on A is a map :

$$f : A\times A \longrightarrow A \qquad (1.2.1a)$$

$$f : (\alpha,\beta) \longrightarrow f(\alpha,\beta)=c \qquad (1.2.1b)$$

Instead of writing f(a,b) we use a symbol, such as $\circ$, $\square$, for denoting the composition law and we write $a\circ b=c$. A set A equiped with a composition law is called **algebraic structure** or **algebraic system** and is denoted by $(A.\circ)$

DEFINITION I.2.2 : A **group** is an algebraic system $(G.\circ)$, where the internal composition law is associative, there exists a unit element e, and every element α possesses an inverve α^{-1}, that is :

$$\text{i)} \qquad (\forall\alpha,\beta,\gamma\in G)\Big[\ (\alpha\circ\beta)\circ\gamma=\alpha\circ(\beta\circ\gamma)\ \Big] \qquad (1.2.2a)$$

$$\text{ii)} \qquad (\exists e\in G)(\forall\alpha\in G)\Big[\ \alpha\circ e=e\circ\alpha=\alpha\ \Big] \qquad (1.2.2b)$$

$$\text{iii)} \qquad (\forall a\in G)(\exists a^{-1}\in G)\Big[\ a\circ a^{-1}=a^{-1}\circ a=e\ \Big] \qquad (1.2.2c)$$

If G is a group with the additional property :

iv) $(\forall a,b\in G)\left[\ a\circ b=b\circ a\ \right]$ (1.2.2d)

then the group G is called **commutative** group or **Abelian group**.

If Γ is a subset of G and obeys the group axioms (i), (ii), and (iii), then Γ is called a **subgroup** of G.

DEFINITION 1.2.3 : A map $f : G \longrightarrow G'$ between two groups $(G,\circ)$ and $(G',\square)$ is called **homomorphism**, if the following property holds :

$$(\forall a,b\in G)\left[\ f(a\circ b)=f(a)\square f(b)\ \right] \qquad (1.2.3)$$

Thus a homomorphism "carries" the composition law $\circ$ on G to the composition law $\square$ on G'. Homomorphisms of groups are well visualized in some important aspects with the help of two concepts, the **image** Im(f) and the **kernel** Ker(f) of the homeomorphism.

DEFINITION 1.2.4 : If $f : G \longrightarrow G'$ is a group homomorphism, then we define :

a) $$\mathrm{Im}(f) = \left\{\ f(\alpha)\ /\ \alpha\in G\ \right\} \qquad (1.2.4)$$

b) $$\mathrm{Ker}(f) = \left\{\ \alpha\in G\ /\ f(\alpha)=e'\in G'\ \right\} \qquad (1.2.5)$$

It is well known that Im(f) is a subgroup of G' and Ker(f)

is a subgroup of G.

DEFINITION I.2.5 : A homomorphism f between two groups G and G' is called **isomorphism** if f is bijective. In the case where $G=G'$ an homomorphism f is called **endomorphism** and an isomorphism is called **automorphism.**

DEFINITION I.2.6 : Let $(G,\circ)$ be a group with unit element e. We define as an **"isogroup"** $\hat{G}$ an isotope of the group G (Santilli, 1978a) equipped now with a new composition law, denoted by $*$ and with a new unit $\hat{I}$, which will be called **isounit**, (which does not need to belong to G), so that the pair $(\hat{G},*)$ verifies all the properties to have the group structure.

Thus an isogroup is a group by construction. A way to define the new composition law is the following :

Firstly, we reconstruct the elements of the group $\hat{G}$ as

$$G \ni o \longrightarrow \hat{o} \equiv o\hat{I} \qquad (I.2.6)$$

where the isounit $\hat{I}$ is defined with the help of an invertible element T : $\hat{I}=T^{-1}$, called **isotopic element**, (fixed or not). The new composition law is defined by (Santilli, loc. cit) :

$$(\forall\hat{o},\hat{\beta}\in\hat{G})\left[\ \hat{o}*\hat{\beta} \equiv \hat{o}T\hat{\beta}\ \right] \qquad (1.2.7)$$

It can be proved easily, that $\hat{G}$, with the above internal composition, can become a group with unit $\hat{I}$.

Example. We may consider the group $(R,.)$ of real numbers with internal composition law given by the usual product. As isounit we consider the imaginary unit i which of course, does not belong to R, and construct the set :

$$\hat{R} = \left\{ \hat{o}=o\hat{I}=oi \ / \ o\in R \right\} \qquad (I.2.8)$$

that is, the set of imaginary numbers. On this set we define a new product, denoted by $*$, via an invertible element T, where $T=\hat{I}^{-1}=\frac{I}{i}=-i$, as follows :

$$(\forall\hat{o},\hat{\beta}\in\hat{R})\left[\ \hat{o}*\hat{\beta}\equiv(o\hat{I})T(\beta\hat{I})=(oi)(-i)(\beta i)=o\beta i=o\beta\hat{I}=\ o\hat{\ }\beta\ \right]$$

$$(I.2.9)$$

It is easy to check that the set of imaginary numbers is close under this new product and the pair $(\hat{R},*)$ has the group structure.

The notions of **isosubgroup**, **isomorphism**, **iso-isomorphism**, etc..can be defined in a similar way as above.

I.3 : ISORINGS :

Groups are algebraic systems with one internal composition law. More complicated (and hence, richer) systems are obtained if we introduce a second internal composition law, which is related to the first.

DEFINITION 1.3.I : We recall that a **ring** $R(+.\cdot)$ is an Abelian group $(R,+)$ with identity element e equipped with a second internal composition law, verifying the properties of associativity and of distributivity over the first law :

$$(\forall\alpha,\beta,\gamma\in R)\Big[\ (\alpha\cdot\beta)\cdot\gamma=\alpha\cdot(\beta\cdot\gamma)\ \Big] \qquad (1.3.1a)$$

$$(\forall\alpha,\beta,\gamma\in R)\Big[\ (\alpha+\beta)\cdot\gamma=(\alpha\cdot\gamma)+(\alpha\cdot\gamma)\ \Big] \qquad (1.3.1b)$$

If, in addition to the above properties, we have :

$$(\forall\alpha,\beta\in R)\Big[\ \alpha\cdot\beta=\beta\cdot\alpha\ \Big] \qquad (1.3.1c)$$

the ring R is called **commutative ring**.
If there exists a unit element 1 for the latter law,

$$(\exists 1\in R)(\forall\alpha\in R)\Big[\ \alpha.1=1.\alpha=\alpha\ \Big] \qquad (1.3.1d)$$

then the ring R is called **a ring with unit**.

DEFINITION I.3.2 : A map f : $R \longrightarrow R'$ between two rings $R(+.\cdot)$ and $R'(\square.\circ)$ is called **homomorphism** if it preserves

both composition laws, i.e, if

$$(\forall \alpha,\beta\in R)\left[\ f(\alpha+\beta)=f(\alpha)\square f(\beta)\ \wedge\ f(\alpha\cdot\beta)=f(\alpha)\circ f(\beta)\ \right] \tag{1.3.4}$$

DEFINITION 1.3.3 : A homomorphism f between two rings R and R' is called **isomorphism** if f is bijective. In the case where $R=R'$, a homomorphism f is called **endomorphism** and an isomorphism is called **automorphism**.

DEFINITION 1.3.4 : If $f : R \longrightarrow R'$ is a ring homomorphism, then we define :

i) $\mathrm{Im}(f) = \left\{ f(\alpha)\ /\ \alpha\in R \right\}$

ii) $\mathrm{Ker}(f) = \left\{ \alpha\in R\ /\ f(\alpha)=e'\in R' \right\}$

where e' is the identity element of the ring R'.

DEFINITION 1.3.5 : Let $(R,+,\cdot)$ be a ring with unit 1. We define as a **"isoring"** $\hat{R}$ an isotope of the ring R, which is the same set as R but equipped now with a new second internal composition law, denoted by $*$ and with an isounit $\hat{1}$, so that the triple $(\hat{R},+,*)$ has the structure of a ring.

Note that the isotopy $R \longrightarrow R'$ is solely referred to the second composition law (which is often called multiplication), and not to the first one (which is often called addition). A more general notion of isotopy may include both the composition law.

In a similar way , as in the case of a group, we can define the notions of **isosubring, isohomomorphism, isoisomorphism**, between two rings, etc.

I.4 ISOFIELDS

DEFINITION I.4.I : We recall that a **field** $\mathbb{F}(+.\cdot)$ is a commutative ring with unit where every element (except zero), is invertible. More precisely a field $\mathbb{F}(+.\cdot)$ is :

1) an abelian group with respect to an internal operation, which is usally denoted with + and called **addition** and

2) is equipped with a second internal operation, denoted with $\cdot$ and called **multiplication**, so that the following rules hold :

i) $$(\forall\alpha,\beta,\gamma\in\mathbb{F})\left[\alpha(\beta\gamma)=(\alpha\beta)\gamma\right] \tag{1.4.1a}$$

ii) $$(\exists 1\in\mathbb{F})(\forall\alpha\in\mathbb{F})\left[\alpha 1=1\alpha=\alpha\right] \tag{1.4.1b}$$

iii) $$(\forall\alpha\in\mathbb{F})(\exists\alpha^{-1}\in\mathbb{F})\left[\ \alpha\alpha^{-1}=\alpha^{-1}\alpha=1\ \right] \tag{1.4.1c}$$

iv) $(\forall o,\beta\in\mathbb{F})\left[o\beta=\beta o\right]$ (1.4.1d)

v) $(\forall o,\beta,\gamma\in\mathbb{F})\left[o(\beta+\gamma)=o\beta+o\gamma\right]$ (1.4.1e)

In the following we shall denote a field for brevity with $\mathbb{F}$ instead of $\mathbb{F}(+,\cdot)$ and assume all fields, unless otherwise stated, of characteristics zero(*), so as to avoid fields with an axiomatic structure different than that currently used in applied physics.

The sets of real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$ constitute fields with respect to the conventional sum and multiplication. However, the next more complex structure, the quaternions numbers $\mathbb{Q}$, does not constitute a field according to Definition 1.4.1, because we lose the property of commutativity. In going now from the quaternions to the next case of Caley numbers we also lose the associativity.

It should be noted, however, that the notion of fields is also defined without the condition of commutativity, in which case the quaternions are indeed field (see, e.g., Albert 1948).

DEFINITION 1.4.2 : Let $\mathbb{F}$ be a field with identities elements 0 and 1 with respect to addition and multiplication respectively. We define as an "**isotope**" $\hat{\mathbb{F}}$ of the field $\mathbb{F}$ with

---

(*) A field $\mathbb{F}$ is said to have characteristic p if there exists a least positive integer p such that $po = \underbrace{o+o+\ldots+o}_{p\text{ times}} = 0\ \forall o\in\mathbb{F}$.

respect to the multiplication (Santilli 1981b, Myung and Santilli 1982b), the abelian group ($\mathbb{F}$,+) equipped with a new multiplication $\alpha * \beta$ and a new multiplicative unit $\hat{I}$, called **"multiplicative isounit"**, which verifies all properties for $\hat{\mathbb{F}}$ to be a field.

Thus, an isofield is a field by construction. The basic isofields are the **real isofields** $\hat{\mathbb{R}}$, i.e, the infinitely possible isotopes $\hat{\mathbb{R}}$ of the field of real numbers $\mathbb{R}$, which can be symbolically written as :

$$\hat{\mathbb{R}} = \left\{ \ \hat{n} \ / \ \hat{n} = n\hat{I}, \quad n \in \mathbb{R} \ , \ \hat{I} \neq 0 \ \right\} \qquad (1.4.2)$$

and their elements $\hat{n}$ are called **isonumbers**. From definition 1.4.1 it is easy to see that the sum of two isonumbers is the conventional one

$$\hat{n}_1 + \hat{n}_2 = (n_1 + n_2)\hat{I} \qquad (1.4.3)$$

To identify the appropriate isoproduct, we have to recall that the isounit $\hat{I}$ must be the right and left isounit of $\hat{\mathbb{R}}$. This can be done exactly in the same way as in the case of isogroups, if we interpret $\hat{I}$ as the inverse of an element T, called **"isotopic element"**.

$$\hat{I} = T^{-1} \qquad (1.4.4)$$

and define the isoproduct as :

$$\hat{n}_1 * \hat{n}_2 \equiv \hat{n}_1 T \hat{n}_2 = (n_1 T^{-1}) T (n_2 T^{-1}) = n_1 n_2 T^{-1} = \widehat{n_1 n_2} \qquad (1.4.5)$$

Then

$$\hat{I}*\hat{n} = \hat{n}*\hat{I} = \hat{n} \quad \forall\ \hat{n}\in\hat{\mathbb{R}} \qquad (1.4.6)$$

as desired.

Note that the isotopic element T need not be necessarily an element of the original field $\mathbb{R}$, since it can be an integra-differential operator. As we shall see, this feature is of fundamental relevance for the applications of the isotopic theory.

Note also that the lifting $I \longrightarrow \hat{I}$ does not imply a change in the numbers used in a given theory. This can be seen in various ways, e.g., from the fact that the isoproduct of an isonumber $\hat{n}$ times a quantity Q coincides with the conventional product :

$$\hat{n}*Q = (nT^{-1})TQ = nQ \qquad (1.4.7)$$

Note finelly, from the arbitraryness of the isotopic element T in isoproduct (1.4.5), that the field of real numbers $\mathbb{R}$ admits an infinite number of different isotopies.

Another field of basic physical relevance is the complex isofield $\hat{\mathbb{C}}$:

$$\hat{\mathbb{C}} = \left\{ \hat{c} \ / \ \hat{c} = c\hat{I} \ , \ c\in\mathbb{C} \ , \ \hat{I}\neq 0 \right\} \qquad (1.4.8)$$

which plays a fundamental role in the operator formulation of the isotopies of this monograph.

An important property of the notion of isofield is

that of permitting the unification of all possible fields (of characteristic zero) into one single, abstract field, say, $\hat{F}$. This unification can be expressed via the following

PROPOSITION 1.4.1 : The infinitely possible isotopies $\hat{\mathbb{R}}$ of the field of real numbers $\mathbb{R}$ contain, as particular cases, all possible fields of characteristics zero (Santilli 1991a).

PROOF : Let $\mathbb{R}_0 = \mathbb{R}1$ be the field of real numbers with the ordinary unit 1. The field of complex numbers $\mathbb{C}$ is an isotope of $\mathbb{R}$ because it can be written as the axiom-preserving tensorial product

$$\mathbb{C} = \hat{\mathbb{R}} = \mathbb{R}1 \otimes \mathbb{R}\hat{1}_1 \ , \quad \hat{1}_1 = i \qquad (1.4.9)$$

(or depending on the viewpoint at hand, as the direct sum $\mathbb{C} = \hat{\mathbb{R}} = \mathbb{R}1 \oplus \mathbb{R}\hat{1}_1$), where i is the conventional imaginary unit. In this case the isounit is the tensorial product $\hat{1} = 1 \otimes \hat{1}_1$, while generic elements have the stucture $n = o \otimes \beta$, $o, \beta \in \mathbb{R}$, so $c = n\hat{1} = (o \otimes \beta)(1 \otimes \hat{1}_1) = o1 \otimes \beta\hat{1}_1 = o + i\beta$.

Another approach to the complex field as an isotope of the real numbers is to introduce a variable isounit of the form $\beta\hat{1}$, $\beta \in \mathbb{R}$ and $\hat{1} = 1 \otimes \hat{1}_1$. In this way every complex number c can be written as :

$$c = \alpha\hat{1} = \alpha(1 \otimes \hat{1}_1) = o \otimes o\beta i = \alpha + o\beta i \qquad (1.4.10)$$

In turn, by relaxing the commutativity condition, the field of quaternions $\mathbb{Q}$ is an isotope of $\mathbb{C}$ and therefore of $\mathbb{R}$, because it can be written as the tensorial product :

$$\mathbb{Q} \equiv \hat{\mathbb{C}} = \hat{\mathbb{R}}' = \mathbb{C}\hat{I}_2 \qquad (1.4.11)$$

where $\hat{I}_2=1\otimes\hat{I}_2$, $\hat{I}_2=j$, $j^2=-1$, and $ij=-ji=k$. Using this notation we have :

$$\mathbb{Q}\ni q = c\hat{I}_2 = c(1\otimes\hat{I}_2) = c_1 1\otimes c_2\hat{I}_2 = (o+i\beta)\otimes(\gamma+j\delta) =$$

$$= \alpha\gamma+\beta\gamma i+o\delta j+\beta\delta ij = \alpha\gamma+\beta\gamma i+o\delta j+\beta\delta k = \alpha'+\beta' i+\gamma' j+\delta' k = q\in\mathbb{Q} \qquad (1.4.12)$$

Note that the isotopy $\mathbb{F} \longrightarrow \hat{\mathbb{F}}$ to be used in this monograph is solely referred to the multiplication, and not to the addition. Needless to say, a more general notion of isotopy including both sum and myltiplication as well as internal and external operations, is conceivable, but its study is left for brevity to the interested reader.

1.5 ISOVECTORSPACES AND ISOTRANSFORMATIONS

Let $(V,+)$ be an additive abelian group of elements $x,y,z,\ldots$and $\mathbb{F}$ a commutative field of elements $\alpha,\beta,\gamma,\ldots$Let us introduce an external operation $\mathbb{F}\times V \longrightarrow V$ by defining the multiplicative action of $\mathbb{F}$ on V. This means that to every pair (α,x) with $o\in\mathbb{F}$ and $x\in V$ we assign a composite $\alpha\circ x$ (to be denoted in the sequel simply by the juxtaposition αx)

such that the following properties hold :

$$(\forall\alpha,\beta\in\mathbb{F})(\forall x\in V)\Big[\ \alpha(\beta x)=(\alpha\beta)x\ \Big] \tag{1.5.1a}$$

$$(\forall x\in V)\Big[\ ex=x\ \Big]\ e\text{ is the additive indentity element of }\mathbb{F}. \tag{1.5.1b}$$

$$(\forall\alpha\in\mathbb{F})(\forall x,y\in V)\Big[\ \alpha(x+y)=\alpha x+\alpha y\ \Big] \tag{1.5.1c}$$

$$(\forall\alpha,\beta\in\mathbb{F})(\forall x\in V)\Big[\ (\alpha+\beta)x=\alpha x+\beta x\ \Big] \tag{1.5.1d}$$

DEFINITION I.5.I :The algebraic system so defined is called a **linear space** or **vector space** over the field $\mathbb{F}$, and denoted by $(V,\mathbb{F},+,\circ)$ or simply V.

DEFINITION 1.5.2 : A subset U of a vector space V is called **vector subspace** if it is a subsystem which obeys the axioms of linear space in itself, that is U is closed under vector addition and scalar multiplication.

DEFINITION I.5.3 : Let V and U two vector spaces over the same field $\mathbb{F}$ (not necessarily of the same dimension). A map $f : V \longrightarrow U$ is called **linear map** or **linear transformation** if the following property is holds :

$$(\forall\alpha,\beta\in\mathbb{F})(\forall x,y\in V)\Big[\ f(\alpha x+\beta y)\ =\ \alpha f(x)+\beta f(y)\ \Big]$$

In case V=U, the map f is called **operator**.

The notions of Ker(f) and Im(f) are defined by the relations

$$\mathrm{Ker}(f) = \Big[\, x\in V \;/\; f(x)=0\in U \,\Big]$$

$$\mathrm{Im}(f) = \Big[\, f(x)\in U \;/\; x\in V \,\Big]$$

It easy proved that Ker(f) Im(f) are subspaces of V and U respectively.

From definition I.5.I one can see that we cannot construct an isotopy of a linear space without first introducing an isotopy of the field, because the multiplicative unit I of the space is that of the underlying field.

DEFINITION I.5.4 Let V be a linear space over the field $\mathbb{F}$ and $\hat{\mathbb{F}}$ an isofield of $\mathbb{F}$. We define as **"isospace" or "isovector space"** (Santilli I983a) the linear space $\hat{V}$, (which is the same set as V), over the isofield $\hat{\mathbb{F}}$ equipped with a new external operation $\odot$ which is such to verify all the axioms for a linear space, i.e,

$$(\forall\hat{o},\hat{\beta}\in\hat{\mathbb{F}})(\forall x\in\hat{V})\Big(\ \hat{o}\odot(\hat{\beta}\odot x) = (\hat{o}\hat{*}\hat{\beta})\odot x\ \Big) \tag{I.5.2a}$$

$$(\forall\hat{o}\in\hat{\mathbb{F}})(\forall x,y\in\hat{V})\Big(\ \hat{\alpha}\odot(x+y) = \hat{\alpha}\odot x+\hat{o}\odot y\ \Big) \tag{I.5.2b}$$

$$(\forall\hat{o},\hat{\beta}\in\hat{\mathbb{F}})(\forall x\in\hat{V})\Big(\ (\hat{o}\hat{+}\hat{\beta})\odot x = \hat{o}\odot x+\hat{\beta}\odot x\ \Big) \tag{I.5.2c}$$

$$(\forall x\in V)\Big(\ \hat{I}\odot x = x\odot\hat{I} = x\ \Big) \tag{I.5.2d}$$

Note the lifting of the field, but the elements of the linear space remain unchanged. This is a property of important physical consequence, inasmuch as it is at the foundation of the preservation of the conventional generators of Lie algebras under isotopies. In turn, this implies the preservation of conventional conservation laws under lifting. From the invariance of the elements x,y,z,... of the space under isotopy the following proposition is obtain :

PROPOSITION 1.5.1 : The basis of a linear space V remains unchanged under isotopy.

Let V and V′ be two linear spaces over the same field $\mathbb{F}$. A **linear transformation** is a map :

$$f : V \longrightarrow V'$$

which preserves both the sum and the scalar multiplication, i.e. :

$$(\forall x,y\in V)(\forall \alpha\in \mathbb{F})\Big(\; f(x+y) = f(x)+f(y) \quad \wedge \; f(\alpha x) = \alpha f(x) \;\Big) \tag{1.5.3}$$

or equivalently :

$$(\forall x,y\in V)(\forall \alpha,\beta\in \mathbb{F})\Big(\; f(\alpha x+\beta y) = \alpha f(x)+\beta f(y) \;\Big) \tag{1.5.4}$$

DEFINITION 1.5.5 : An **isolinear transformation** is an isomap :

$$\hat{f} : \hat{V} \longrightarrow \hat{V}'$$

between two isolinear vector spaces $\hat{V}$ and $\hat{V}'$ over the same

isofield $\hat{F}$ which preserves the sum and isomultiplication, i.e., which is such that (Santilli 1991a)

$$(\forall x,y\in V)(\forall\hat{\alpha},\hat{\beta}\in\hat{F})\left(\hat{f}(\hat{\alpha}*x+\hat{\beta}*y)=\hat{\alpha}*\hat{f}(x)+\hat{\beta}*\hat{f}(y)\right) \quad (1.5.5)$$

In physical applications, the spaces V and V′ are usually assumed to coincide, V≡V′, in which case the linear map f is an endomorphism, (usually called **linear operator**), and is realized by the familiar right modular-associative type :

$$x' = f(x) \equiv \mathbf{A}x \quad \forall x\in V \quad (1.5.6)$$

where **A** is independent from the local variables and the product **A**x is associative. A similar notion would evidently result for a left modular associative action : $x'=x\mathbf{A}$.

The transformation is nonlinear when it has the form :

$$x' = \mathbf{A}(x)x \quad (1.5.7)$$

i.e., when **A** has an explicit dependence in the local coordinates x. If the x-dependence is of integral type, we shall say that the above transformation is **nonlocal**.

Assume now that the isospaces $\hat{V}$ and $\hat{V}'$ coincide : $\hat{V}\equiv\hat{V}'$. Then the isotransformation $\hat{f}$ can be realized through the right modular, associative isotopic action :

$$x' = \hat{f}(x) = \mathbf{A}*x = \mathbf{A}Tx \quad T=\text{fixed} \quad (1.5.8)$$

where the action $\mathbf{A}*x$ is still associative. A similar notion would result for a left, modular-isotopic action :

$$x' = \hat{f}(x) = x*\mathbf{A} = xT\mathbf{A} \quad (1.5.9)$$

REMARK I.5.I : When a transformation f is realized by the element **A**, the isotransformation $\hat{f}$ is realized by the element **A**T, where $T^{-1}=\hat{I}$ ($\hat{I}$=isounit of the isofield $\hat{F}$).

DEFINITION I.5.6 : An isotransformation $\hat{f}$ is said to be **isolinear** and/or **isolocal** when the element **A** , (that is f) is conventionally linear and/or local, respectively, i.e., when all nonlinear and/or nonlocal terms are embedded in the isotopic element T.

A number of properties of isotransformations can be easily proved. At the level of abstract axioms, all distinctions between the ordinary multiplication xy and the isotopic one x*y (transformations **A**x and **A***x) cease to exist, in which case linear and isolinear spaces (linear and isolinear transformations) coincide.

However, the isotopies are nontrivial, as illustrated by a number of properties. First, one readily prove the following :

PROPOSITION I.5.2 : Conventional linear transformations f on an isolinear space $\hat{V}$ violate the conditions of isolinearity.

Explicitly stated, the lifting of the Euclidean spaces and of the Minkowski spaces into their corresponding isospaces requires the necessary abandonment, for mathematical consistency, of the Galilean and Lorentz transformations in favor of suitable isolinear and isolocal

generalizations.

PROPOSITION I.5.3 : A isotransformation $\hat{f}$ which is isolinear and isolocal in an isospace $\hat{V}$ is generally nonlinear and nonlocal in V.

In fact, when explicitly written out, isotransformations (1.5.8) become :

$$x' = \mathbf{A}Tx = \mathbf{A}T(t,x,\dot{x},\ddot{x},\ldots)x \qquad (1.5.10)$$

the nonlinearity and nonlocality of the transformations then becomes evidently dependent on the assumed explicitly form of the isotopic element T.

Another simple but important property is the following :

PROPOSITION I.5.4 : Under sufficient topological conditions, nonlinear transformations on a linear vector space **V** can always be cast into an equivalent isolinear form on an isospace $\hat{V}$.

In different terms, given a map f in **V** which violates the conditions of linearity and/or of locality, there always exist an isotope $\hat{V}$ of V under which $\hat{f}$ is isolinear and/or isolocal. Explicitly, nonlinear transformations (6.1.2) can always be written :

$$x' = \mathbf{A}(x)x \equiv \mathbf{B}T(x)x = \mathbf{B}{*}x \qquad (1.5.11)$$

i.e., for **A**=**B**T, with **B** linear.

The above property has important mathematical and

physical implications. On mathematical grounds we learn that *nonlinearity and nonlocality are mathematical characteristics without an essential axiomatic structure, because they can be made to diasappear at the abstract level via isotopic liftings.*

In turn, this feature is not a mere mathamatical curiosity, but has a number of possible mathematical applications. As an example, if properly developed, the isotopies of the current theory of linear equations may be assistance in solving equivalent nonlinear systems.

On physical grounds, the first application of the notions presented in this section is that of rendering more manageable the formulation and treatment of nonlinear nad nonlocal generalizations of Galilean or Lorentzian theories which if treated conventionally, are of a notoriously difficult (if not impossible) treatment.

The physical implications are however deeper than that. Recall that the electromagnetic interactions have been fully treatable with linear and local theories, such as the symmetry under the conventional Lorentz transformations.

One of the central open problem of contemporary theoretical physics (as well as of applied mathematics) is the still unanswered, historical legacy by Fermi (1949) and other Founders of contemporary physics on the ultimate nonlinearity and nonlocality of the strong interactions.

All attempts conducted until now in achieving a nonlinear and nonlocal extension of current theories via conventional techniques have met with rather serious problem

of mathematical consistency and/or physical effectiveness, as well known.

Because of their simplicity, the isotopies appear to have all the necessary ingredients for the achievement of a mathematically consistent and physically effective nonlinear and nonlocal generalization of the current theories for the electromagnetic interactions via the mere generalization of the trivial unity I into the isounit $\hat{I}$, and the consequential isotopic generalization of the various notions of field, spaces, transformations, etc.

The mathematical consistency of the isotopies is self-evident from their simplicity. Their physical effectiveness is due to the fact that, given a linear theory, say a Hamiltonian description of a conservative trajectory on a metric space, all the possible nonlinear and nonlocal generalizations are guaranteed by the mere isotopies of the underlying space.

1.6 ISOMODULE

As well known, a generalization of the notion of vector space over a field is provided by a **module over a ring** R with unit.

DEFINITION 1.6.1 : As **R-module** one means an additive group M, together with a map $R \times M \longrightarrow M$, verifying properties (1.5.1a) - (1.5.1d).

DEFINITION I.6.2 : Let M to be an R-module and $\hat{R}$ an isoring of R. We define as **iso-R-module** the $\hat{R}$-module $\hat{M}$ (which is the same set as M), equipped with a new external operation $\odot$, which is such to preserve the axioms (1.5.2a)-(1.5.2d) for a R-module.

The notion of isomodule was introduced for the first time by Santilli (1979c).

I.7 ISOSPACES

In order to lift other important mathematical structures, as metric space, Banach space, inner product space (Hilbert space), etc, we have firstly to define the notion of isobilinear form.

We recall that given a linear space V over the field $\mathbb{F}$, a bilinear form is a map f :

$$f : V{\times}V \longrightarrow \mathbb{F}$$

with the folowing property :

$$(\forall \alpha,\beta\in\mathbb{F})(\forall x,y,z\in V)\Big(f(\alpha x+\beta y,z)=\alpha f(x,z)+\beta f(y,z) \wedge$$
$$f(x,\alpha y+\beta z)=\alpha f(x,y)+\beta f(x,z) \Big) \qquad (1.7.1)$$

An isotope $\hat{f}$ of this bilinear form f, which is here called **isobilinear form**, can be defined if we lift the field $\mathbb{F}$ to an isofield $\hat{\mathbb{F}}$, such that

$$(\forall\hat{\alpha},\hat{\beta}\in\hat{\mathbb{F}})(\forall x,y,z\in V)\Big(\ f(\hat{\alpha}\odot x+\hat{\beta}\odot y,z)=\hat{\alpha}*f(x,z)+\hat{\beta}*f(y,z)\ \wedge$$

$$f(x,\hat{\alpha}\odot y+\hat{\beta}\odot z)=\hat{\alpha}*f(x,y)+\hat{\beta}\odot f(x,z)\ \Big) \qquad (1.7.2)$$

Let $\{e_i\}$ $i=1,\dots,n$ be a basis of V, where n=dimV. Then, as is well known, a bilinear form f is represented by n^2 constants $g^{ij}=f(e_i,e_j)$ as follows :

$$\text{If } x=\sum_i x_ie_i \text{ and } y=\sum_j y_je_j \text{ then } f(x,y)=\sum_{i,j} x_iy_jf(e_i,e_j)=$$

$$=\sum_{i,j} x_i g^{ij}y_j \qquad (1.7.3)$$

As a consequence, the isobilinear form $\hat{f}$ can be represented by n^2 isonumbers $\hat{g}^{ij}=\hat{f}(e_i,e_j)$.

In the case of an **inner product space** $(V,<,>)$, the bilinear form <,>, which is called **inner product**, is a map of the form $V\times V \longrightarrow \mathbb{F}$, with the properties :

$$(\forall x\in V)\Big(\ <x,x>\geq 0 \text{ and } <x,x>=0 \text{ iff } x=0\ \Big) \qquad (1.7.4a)$$

$$(\forall x,y\in V)\Big(\ <x,y>=\overline{<y,x>}\ \Big) \qquad (1.7.4b)$$

$$(\forall \alpha,\beta\in\mathbb{F})(\forall x,y,z\in V)\Big(\ <x,\alpha y+\beta z>=\alpha<x,y>+\beta<x,z>\ \Big) \qquad (1.7.4c)$$

An **isoinner product space** $\hat{V}$ is defined as the isospace V, over $\hat{\mathbb{F}}$, equipped with an **isoinner product** $\hat{<,>}$ with

similar properties :

$$(\forall x\in V)\Big(\ \langle x\hat{,}x\rangle\geq 0 \text{ and } \langle x\hat{,}x\rangle=0 \text{ iff } x=0\ \Big) \qquad (1.7.5a)$$

$$(\forall x,y\in V)\Big(\ \langle x\hat{,}y\rangle=\overline{\langle y\hat{,}x\rangle}\ \Big) \qquad (1.7.5)$$

$$(\forall\hat{\alpha},\hat{\beta}\in\mathbb{F})(\forall x,y,z\in V)\Big(\ \langle x\hat{,}\hat{\alpha}\odot x+\hat{\beta}\odot z\rangle=\hat{\alpha}*\langle x\hat{,}y\rangle+\hat{\beta}*\langle x\hat{,}z\rangle\ \Big) \qquad (1.7.5c)$$

Let us now recall the general notion of metric space. If M is a non-empty set and d a function :

$$d : M\times M \longrightarrow \mathbb{R}$$

which obeys the axioms :

i) $(\forall x,y\in M)\Big(\ d(x,y)\geq 0 \text{ and } d(x,y)=0 \text{ iff } x=y\ \Big)$ (1.7.6a)

ii) $(\forall x,y\in M)\Big(\ d(x,y)=d(y,x)\ \Big)$ (1.7.6b)

iii) $(\forall x,y,z\in M)\Big(\ d(x,y)\leq d(x,z)+d(z,y)\ \Big)$ (1.7.6c)

then the function d is called a **metric** on M and the pair (M,d), **metric space**. If the axiom (i) is replace by

$$(\forall x,y\in M)\Big(\ d(x,y)\geq 0 \text{ and } d(x,x)=0\ \Big) \qquad (1.7.7)$$

the metric space M is called **pseudometric space**.

If now the metric space M has the structure of an n-dimentional linear vector space and $\{e_i\}$ is a base of M, then the familiar way of realizing the function d , which is assumed to be a bilinear form, is via the n^2 constant :

$$g^{ij}=d(e_i,e_j).$$

So if $x=\sum_i x_i e_i$ and $y=\sum_j y_j e_j$, then

$$d(x,y) = d\Big(\sum_i x_i e_i, \sum_j y_j e_j\Big) = \sum_{i,j} x_i y_j d(e_i,e_j) =$$

$$= \sum_{i,j} x_i g^{ij} y_j \qquad (1.7.8)$$

The best physical example of a metric space is the n-dimentional Euclidean space hereon denoted with the symbol $E(r,\delta,\mathbb{R})$, where the metric δ is determined by the Kronecker delta δ^{ij} :

$$\delta = (\delta^{ij}) = \mathrm{diag}(1,1,\dots,1)$$

A pseudometric space of primary physical relevance is the (3+1)-dimensional **Minkowski space**, hereon denoted $M(x,\eta,\mathbb{R})$, whose the elements are of the form :

$$x = (x_\mu) = (x_i x_4) \ , \ x_i \in E(r,\delta,\mathbb{R}) \ , \ x_4 = c_0 t$$

where $c_0 \in \mathbb{R}$ represents the speed of light in vacuum. The metric is indefinite and defined by :

$$\eta(x,y) = x_\mu \eta^{\mu\nu} y_\nu \gtreqless 0 \qquad (1.7.9)$$

which is the well known Minkowski metric of the type :

$$\eta = \mathrm{diag.}(1,1,1,-1) \qquad (1.7.10)$$

Futher spaces also relevant in physics are the Riemannian spaces hereon denoted $R(a,g,\mathbb{F})$.

In 1983 Santilli (1983b) constructed for the first time the isotopes $\hat{M}(x,\hat{g},\hat{\mathbb{F}})$ of $M(x,g,\mathbb{F})$ which are playing an increasingly fundamental role in physics. Santilli's construction of the isospaces can be formulated by introducing the n-dimentional, nowhere null and Hermitean isounits

$$\hat{I} = (I_i^{\ j}) = (1^j_{\ i}) \qquad i,j=1,2,\ldots,n \qquad (1.7.11)$$

with isotopic elements

$$T = \hat{I}^{-1} = (T_i^{\ j}) \qquad (1.7.12)$$

Then we can introduce the **isomap**

$$\hat{g}(x,y) = (x_i \hat{g}^{ij} y_j)\hat{I} \qquad (1.7.13)$$

where the quantity

$$\hat{g} = Tg = (T_j^{\ k} g^{kj}) \qquad (1.7.14)$$

is called hereon the **isometric**.

The basis $e=\{e_i\}$, $i=1,2,\ldots,n$ of an n-dimentional space $M(x,g,\mathbb{F})$ can be defined via the rules

$$\hat{g}(e_i,e_j) = g_{ij} \qquad (1.7.15)$$

The above isotopic generalizations can be expressed as

follows :

DEFINITION I.7.I : The "isotopic liftings" of a given n-dimentional metric or pseudometric space $M(x,g,\mathbb{F})$ are given by the infinitely possible "isotopes" $\hat{M}(x,\hat{g},\hat{\mathbb{F}})$ characterized by :

a) the same dimention n and the same local coordinates x of the original space,

b) the isotopies of the original metric g into one of the infinitely possible nonsingular Hermitean "isometrics" $\hat{g}=Tg$ on $\mathbb{F}$ with isotopic element T depending on the local variables x, their derivatives $\dot{x},\ddot{x},..$ with respect to an indepedent parameter, as well as any needed additional quantity :

$$g \longrightarrow \hat{g}=Tg \qquad (1.7.16)$$

$T=T(x,\dot{x},\ddot{x},...)$, $\det T\neq 0$, $T^{2}=T$, $\det g\neq 0$, $g=g^{2}$, and

c) the lifting of the field $\mathbb{F}$ into the isotope $\hat{\mathbb{F}}$, whose isounit $\hat{1}$ is the inverse of the isotopic element T, i.e. :

$$\hat{\mathbb{F}}=\mathbb{F}\hat{1}, \quad \hat{1}=T^{-1}=\hat{\delta}^{-1} \qquad (1.7.17)$$

with composition now in $\hat{\mathbb{F}}$

$$\hat{g}(x,y)=(x\hat{,}y)=(x,Ty)\hat{1}=(Tx,y)\hat{1}=\hat{1}(x,Ty)=\sum_{i,j}^{n}(x^{i}\hat{g}_{ij}y^{j})\hat{1} \qquad (1.7.18)$$

The following three cases are particularly important

(Santilli I983b; 1991a) :

The liftings of the conventional n-dimentional Euclidean spaces $E(r,\delta,\mathbb{R})$ over the reals $\mathbb{R}$ into "Euclidean-Santilli spaces" or "Santilli's isoeuclidean spaces", are given by :

$$E(r,\delta,\mathbb{R}) \longrightarrow \hat{E}(r,\hat{\delta},\hat{\mathbb{R}}) \qquad (I.7.I9a)$$

$$\delta = I_{nxn} \longrightarrow \hat{\delta} = T(r,\dot{r},\ddot{r},\ldots)\delta \qquad (I.7.I9b)$$

$$\det.T \neq 0 \ , \ T = T^{\dagger} \ , \ \det.\hat{\delta} \neq 0 \ , \ \hat{\delta}^{\dagger} = \hat{\delta} \qquad (I.7.I9c)$$

$$\mathbb{R} \longrightarrow \hat{\mathbb{R}} = \mathbb{R}\hat{I} \ , \ \hat{I} = T^{-1} = \hat{\delta}^{-1} \qquad (I.7.I9d)$$

$$\delta(r_1,r_2) = \sum_{i,j}^{n} r_1^i \delta_{ij} r_2^j \longrightarrow \hat{\delta}(r_1,r_2) =$$

$$(r_1,\hat{\delta}r_2)\hat{I} = (\hat{\delta}r_1,r_2)\hat{I} = \hat{I}(r_1,\hat{\delta}r_2) = \sum_{i,j}^{n} r_1^i \hat{\delta}_{ij} r^j \hat{I} \qquad (1.7.I9e)$$

The liftings of the conventional Minkowski space $M(x,\eta,\mathbb{R})$ in (3+I)-space-time dimensions are given by the isotopes called "Minkowski-Santilli spaces" or "Santilli's isominkowski spaces".

$$M(x,\eta,\mathbb{R}) \longrightarrow \hat{M}(x,\hat{\eta},\hat{\mathbb{R}}) \qquad (1.7.20a)$$

$$\eta = \mathrm{diag.}(I,I,1,-1) \longrightarrow \hat{\eta} = T(x,\dot{x},\ddot{x},\ldots)\eta \qquad (I.7.20b)$$

$$\det.T \neq 0 \ , \ T = T^{\dagger} \ , \ \det.\hat{\eta} \neq 0 \ , \ \hat{\eta}^{\dagger} = \hat{\eta} \qquad (I.7.20c)$$

$$\mathbb{R} \longrightarrow \hat{\mathbb{R}}=\mathbb{R}\hat{1} \quad , \quad \hat{1}=T^{-1} \tag{1.7.20d}$$

$$(x,y)=\sum_{i,j}^{n} x^{\mu}\eta_{\mu\nu}y^{\nu} \longrightarrow (x\hat{,}y)=(x,Ty)\hat{1}=(Tx,y)\hat{1}=$$

$$=\hat{1}(x,Ty)=\sum_{\mu,\nu}^{n} x^{\mu}\eta_{\mu\nu}y^{\nu}\ \hat{1} \tag{1.7.20e}$$

Finally, the liftings of a given n-dimensional, Riemannian or pseudoriemannian space $R(x,g,\mathbb{R})$ over the reals $\mathbb{R}$ into the infinitely possible "Riemannian-Santilli spaces" or "Santilli's isoriemannian spaces" $\hat{R}(x,\hat{g},\hat{\mathbb{R}})$ are given by

$$R(x,g,\mathbb{R}) \longrightarrow \hat{R}(x,\hat{g},\hat{\mathbb{R}}) \tag{1.7.21a}$$

$$g \longrightarrow \hat{g}=T(x,\dot{x},\ddot{x},\ldots)g \tag{1.7.21b}$$

$$\det.g\neq 0 \quad , \quad g=g^{t} \quad , \quad \det.T\neq 0 \quad , \quad T=T^{t} \quad , \quad \det.\hat{g}\neq 0 \quad , \quad \hat{g}=\hat{g}^{t} \tag{1.7.21c}$$

$$\mathbb{R} \longrightarrow \hat{\mathbb{R}}=\mathbb{R}\hat{1} \quad , \quad \hat{1}=T^{-1} \tag{1.7.21d}$$

$$(x,y)=\sum_{i,j}^{n} x^{i}g_{ij}y^{j} \longrightarrow (x\hat{,}y)=(x,Ty)\hat{1}=((Tx,y)\hat{1}=\hat{1}(x,Ty)=$$

$$=\sum_{i,j}^{n} x^{i}\hat{g}_{ij}y^{j} \tag{1.7.21e}$$

The general character of the concept of isotopy is illustrated by the following property proved in the original proposal (SAntilli 1983b).

PROPOSITION 1.7.2 : All possible metric and pseudometric spaces in n-dimension $M(r,g,\mathbb{F})$ can be interpreted as isotopes of the Euclidean space in the same dimension $E(r,\delta,\mathbb{F})$

$$\hat{M}(r,g,\hat{\mathbb{F}}) : \hat{\mathbb{F}}=\mathbb{F}\hat{I} , \hat{I}=g^{-1} \qquad (1.7.22)$$

The following Corollary illustrates the fact that there is no need to study the isotopies of all spaces, because those of the fundamental Euclidean space are sufficient.

COROLLARY 1.7.1: The conventional Minkowski space $M(x,\eta,\mathbb{R})$ in (3+1)-space-time dimensions over the reals $\mathbb{R}$ can be interpreted as an isotope $\hat{M}(x,\hat{\delta},\hat{\mathbb{R}})$ of the 4-dimentional Euclidean space $E(x,\delta,\mathbb{R})$ characterized by the isotopy of the metric :

$$\delta=I_{4x4}=\hat{\delta}=T\delta=\eta=\text{diag.}(1,1,1,-1) \qquad (1.7.23)$$

under the redefinition of the fields :

$$\mathbb{R} \longrightarrow \hat{\mathbb{R}}=\mathbb{R}\hat{I} , \quad \hat{I}=T^{-1}=\eta^{-1}=\eta \qquad (1.7.24)$$

The reader should remember that the isotopy of the field is a feature needed for mathematical consistency, which however does not affect the practical numbers of the theory owing to the property $\hat{N}*x\equiv Nx$, $\hat{N}\in\hat{\mathbb{R}}$, $x\in M$. Also, the symmetries of $\hat{M}(x,\eta,\hat{\mathbb{R}})$ and those of $M(x,\eta,\mathbb{R})$ coincide because characterized by the metric η. Thus, the Minkowski-Santilli space $\hat{M}(x,\eta,\hat{\mathbb{R}})$ and the conventional Minkowski space $M(x,\eta,\mathbb{R})$ can be made to coincide for all practical

purposes .

COROLLARY 1.7.2 :The conventional Riemannian spaces $R(x,g,\mathbb{R})$ in (3+1)-space-time dimensions over the reals $\mathbb{R}$ is an isotope $\hat{R}(x,g,\hat{\mathbb{R}})$ of the 4-dimensional Euclidean space $E(x,\delta,\mathbb{R})$ characterized by the lifting of the Euclidean metric δ into the Riemannian metric g :

$$\delta=I_{4x4} \longrightarrow T\delta=g \qquad (1.7.25)$$

and by the corresponding lifting of the field :

$$\mathbb{R} \longrightarrow \hat{\mathbb{R}}=\mathbb{R}\hat{I} \;, \quad \hat{I}=T^{-1}=g^{-1} \qquad (1.7.26)$$

We also have the following alternative interpretation of the Riemannian space.

COROLLARY 1.7.3 : The conventional Riemannian space $R(x,g,\mathbb{R})$ in (3+1)-space-time dimensions over the reals $\mathbb{R}$ can be interpreted as an isotope $\hat{R}(x,g,\hat{\mathbb{R}})$ of the Minkowski space $M(x,\eta,\mathbb{R})$ in the same dimension characterized by the isotopy of the Minkowski metric :

$$\eta=\text{diag.}(1,1,1,-1) \longrightarrow T(x)\eta=g(x) \qquad ((1.7.27)$$

and of the field :

$$\mathbb{R} \longrightarrow \hat{\mathbb{R}}=\mathbb{R}\hat{I} \;, \quad \hat{I}=T^{-1} \qquad (I.7.28)$$

The notion of isotopy of a metric or pseudometric space is therefore first useful for conventional formulations. In fact, Santilli (1991a) proved that the transition from relativistic to gravitational aspects is an isotopy.

This concept is at the foundations of the global symmetries of conventional gravitational theories, which can be readily studied via the Lie-isotopic theory, but which is of otherwise rather difficult treatment via conventional techniques.

Notice also the chain of isotopies illustrated by the above Corollaries, also called **isotopies of isotopies.**

$$E(x,\delta,\mathbb{R}) \longrightarrow M(x,\eta,\mathbb{R}) \longrightarrow R(x,g,\mathbb{R}) \qquad (1.7.29)$$

Corollary 1.5.3 is usefull to illustrate the insensitively of the isotopies to the explicit functional depedence of the isounit. We can then begin to see the vastity of the Euclidean-Santilli spaces, which encompass, not only the Minkowski and Riemannian space, but also all known metric and pseudometric spaces of the same dimension, such as Finslerian spaces, etc., as well as additional classes of infinitely possible, genuine isotopies of the Euclidean, Minkowski, Riemannian and other spaces.

DEFINITION 1.7.2 : Given a metric or pseudometric space $M(x,g,\mathbb{F})$ with metric g, **"Santilli-isodual"** space $\hat{M}^d(x,\hat{g},\hat{\mathbb{F}})$ is the isotopic space $\hat{M}$ characterized by the isotopic element (Santilli 1985b) :

$$T=-I=\text{diag.}(-1,-1,-1,\dots,-1) \qquad (1.7.30)$$

The isodual of the Euclidean space $E(x,\delta,\mathbb{R})$ is therefore the isotope $\hat{E}^d(x,\hat{\delta},\hat{\mathbb{R}})$, where the isometric is given by

$$\hat{\delta} = -\delta \qquad (1.7.31)$$

which cannot be reinterpreted via the inversion, i.e.

$$\sum_{i,j}^{n} x^{i}\hat{g}_{ij}x^{j} \neq \sum_{i,j}^{n} x'_{i}\delta^{ij}x'_{j} \ , \quad x'=-x$$

thus illustrating the independence of Santilli's isoduality

$$E(x,\delta,\mathbb{R}) \longrightarrow \hat{E}^{d}(x,\hat{\delta},\hat{\mathbb{R}})$$

from the inversion $x \longrightarrow -x$.

The above spaces are useful for the construction of the new realizations of given simple Lie groups precisely of isodual type with rather intriguing implications.

Similarly, the isodual of the Minkowski space $M(x,\eta,\mathbb{R})$ is the isospace $\hat{M}^{d}(x,\hat{\eta},\hat{\mathbb{R}})$ where the isometric $\hat{\eta}$ is given by :

$$\hat{\eta}=T\eta=-\eta=\mathrm{diag.}(-1,-1,-1,+1) \qquad (1.7.32)$$

Since we have the joint lifting $\mathbb{R} \longrightarrow \hat{\mathbb{R}}=-\mathbb{R}$, one can see that isoduality implies the mapping

$$x^{\mu}\eta_{\mu\nu}x^{\nu} \longrightarrow -x^{\mu}\eta_{\mu\nu}x^{\nu}$$

while preserving the space-like or time-like character of a vector.

More generally, the <u>same</u> equations of motion are admitted by both, a given space and its isodual. This has led Santilli (Santilli 1991d), to the formulation of a new universal invariance of physical laws under isoduality.

1.8 ISOALGEBRAS

A **linear algebra**, or **algebra** for short, is an algebraic system with two internal and one external composition laws. Such algebraic system combines the features of a ring and of a vector space (Roman 1975a).

DEFINITION 1.8.1 : A **linear algebra** is an algebraic system $(A,\mathbb{F},.)$, where A is a ring, $\mathbb{F}$ is a field, such that the following properties hold :

$$(\forall x,y,z\in A)\left[\ x+(y+z)=(x+y)+z\ \right] \tag{1.8.1a}$$

$$(\exists 0\in A)(\forall x\in A)\left[\ x+0=0+x=x\ \right] \tag{1.8.1b}$$

$$(\forall x\in A)(\exists -x\in A)\left[\ x+(-x)=(-x)+x=0\ \right] \tag{1.8.1c}$$

$$(\forall x,y\in A)\left[\ x+y=y+x\ \right] \tag{1.8.1d}$$

$$(\forall x,y,z\in A)\left[\ x(yz)=(xy)z\ \right] \tag{1.8.2a}$$

$$(\forall x,y,z\in A)\left[\ x(y+z)=xy+xz \wedge (x+y)z=xz+yz\ \right] \tag{1.8.2b}$$

$$(\forall x\in A)(\forall \alpha,\beta\in\mathbb{F})\left[\ \alpha(\beta x)=(\alpha\beta)x\ \right] \tag{1.8.3a}$$

$$(\exists 1\in\mathbb{F})(\forall x\in A)\Big[\ 1x=x1=x\ \Big] \tag{1.8.3b}$$

$$(\forall x,y\in A)(\forall \alpha\in\mathbb{F})\Big[\ \alpha(x+y)=\alpha x+\alpha y\ \Big] \tag{1.8.4a}$$

$$(\forall x\in A)(\forall \alpha,\beta\in\mathbb{F})\Big[\ (\alpha+\beta)x=\alpha x+\beta x\ \Big] \tag{1.8.4b}$$

$$(\forall x,y\in A)(\forall \alpha\in\mathbb{F})\Big[\ \alpha(xy)=(\alpha x)y=x(\alpha y)\ \Big] \tag{1.8.4c}$$

where we use the symbol + to denote both the sum of ring elements $x,y,\ldots$ as well as the sum of elements $\alpha,\beta,\ldots$ of the field $\mathbb{F}$. Similarly, we use simple juxtaposition of symbols to indicate a product xy in the ring, a product $\alpha\beta$ of two scalars, and a product αx of a scalar and of an element of A.

DEFINITION 1.8.2 : If the algebra A, considered as **a** ring, is commutative, i.e.

$$(\forall x,y\in A)\Big[\ xy=yx\ \Big] \tag{1.8.5}$$

then we call A a **commutative algebra**.

If, as a ring, A has a unit element ε, i.e.

$$(\exists \varepsilon\in A)(\forall x\in A)\Big[\ \varepsilon x=x\varepsilon=x\ \Big] \tag{1.8.6}$$

we call A an **algebra with unit**.

Finally, if, as a ring, A is a field, i.e., if it has a unit and

$$(\forall x\in A)(\exists x^{-1}\in A)\left[\ xx^{-1}=x^{-1}x=\varepsilon\ \right] \tag{I.8.7}$$

then we say that A is a **division algebra**.

DEFINITION I.8.3 : A subset V of an algebra A is called a **subalgebra** if it satisfies the algebra axioms, i.e., if it is an algebra in its own right.

It is easy to proove that a subset $V\subset A$ of an algebra A is a subalgebra iff :

$$(\forall x,y\in A)(\forall \alpha\in\mathbb{F})\left[\ x+y\in A\ ,\ \alpha x\in A\ ,\ xy\in A\ \right] \tag{I.8.8}$$

that is, the subset V is closed under all the composition laws.

In the following we discuss some tools, which simplify the handling of algebras.

Let A be an algebra and $\{e_1,\dots,e_n\}$ a basis. If x and y are any two elements of A, we have :

$$x=\sum_i^n \alpha_i e_i \quad\text{and}\quad y=\sum_j^n \beta e_j \tag{1.8.9}$$

and in view of the axioms for algebras, we have :

$$xy=\sum_{ij}^n \alpha_i\beta_j(e_ie_j) \tag{I.8.10}$$

Since $e_ie_j\in A$ we can expand this element with respect to the

same basis :

$$e_i e_j = \sum_k^n c_{ijk} e_k \tag{1.8.11}$$

The coefficients c_{ijk} are called the **structure constants** of the algebra A. They determine uniquely the product of arbitrary elements, since

$$xy = \sum_{ijk}^n \alpha_i \beta_j c_{ijk} e_k \tag{1.8.12}$$

Thus the multiplicative structure of an algebra is fully characterized by the structure constants. It is obvious that there are n^3 structure constants for an algebra of dimensions n. However, they are not independent. The associativity of products gives the constraints :

$$e_i(e_j e_k) = (e_i e_j)e_k \tag{1.8.13}$$

for any three basis elements, which under the equations (1.8.11), give the relations :

$$\sum_k^n c_{ijk} c_{klm} = \sum_k^n c_{ikm} c_{jlm} \tag{1.8.14}$$

DEFINITION 1.8.4 : A map $f : A \longrightarrow A'$ between algebras over the same field $\mathbb{F}$ is called **morphism** if :

$$(\forall x,y \in A)(\forall \alpha,\beta \in \mathbb{F})\Big[f(\alpha x+\beta y)=\alpha f(x)+\beta f(y) \wedge f(xy)=f(x)f(y)\Big] \tag{1.8.15}$$

The usual statements about the image and kernel of a

morphism hold and need not be repeated.

Algebras play a fundamental role in physics, as well known, and their use is predictably enlarged by the isotopies. Among the existing large number of algebras, a true understanding of the formulations presented here, as well as of their operator image, requires a knowledge of the following primary algebras :

DEFINITION 1.8.5 : A **nonassociative algebra** is an algebraic system which obeys all axioms of a linear algebra except (1.8.2a) .

Completely general nonassociative algebras are of little interest. If however, one imposes some additional identities to be satisfied by the elements of a nonassociative algebra, then several types of useful systems, with a rich theory and many applications, ensue.

It should be clear that all concepts, properties, and theorems relating to associative algebras which are independent of the associative law, have their complete counterpart in any nonassociative algebra. Thus, without any further explanation we may speak of dimension, subalgebras, direct sums, ideals, quotient algebra and morphisms of nonassociative algebras.

The most familiar class of nonassociative algebras is known by the name of Lie algebras.

DEFINITION 1.8.6 : A **Lie algebra** is a nonassociative algebra L over the field $\mathbb{F}$ where the product (called **Lie product**), satisfies the following two axioms :

$$(\forall x \in L)\Big[\ xx=0\ \Big] \tag{1.8.17}$$

$$(\forall x,y,z \in L)\Big[\ x(yz)+y(zx)+z(xy)=0\ \Big] \tag{1.8.18}$$

If the characteristic p of the field $\mathbb{F}$ is not 2, then the axiom (1.8.17) is equivalent to the following :

$$(\forall x,y \in L)\Big[\ xy+yx=0\ \Big] \tag{1.8.17a}$$

Property (1.8.17a) of a Lie product is called **anti-symmetry** and property (1.8.18) is known as the **Jacobi identity**, which can be written as

$$y(zx) = (xy)z - x(yz) \tag{1.8.19}$$

The presence of the l.h.s shows that, in general, a Lie product cannot be associative. The r.h.s is called **associator** of the three elements x,y,z and is denoted by :

$$[x,y,z] = (xy)z - x(yz) \tag{1.8.20}$$

A Lie algebra can be characterized in terms of its structure constants by the relations :

$$c_{ikl} = -c_{kil} \tag{1.8.21}$$

$$\sum_{r} \{c_{ijr}c_{rks} + c_{jkr}c_{ris} + c_{kir}c_{rjs}\} = 0 \tag{1.8.22}$$

A familiar realization of the Lie product is given in terms of an associative product xy (where x,y belong to an associative algebra *A*), by the relation :

$$[x,y] = xy-yx \tag{1.8.23}$$

The r.h.s of (1.8.23) is usually called the **commutator** of the elements x,y of the associative algebra *A*. The Lie algebra with the Lie product being defined as the commutator, is called the **commutator algebra of the associative algebra** *A*, or the **attached algebra** and will be denoted by A^{-}. The algebras *A* and A^{-} are identical as linear spaces. The associative algebras *A* are the fundamental Lie-admissible algebras,as we shall see in next chapter.

As we saw, every associative algebra *A* gives rise to a Lie algebra A^{-}. The answer to the converse relation is given by the fundamental result of Poincaré-Birkhoff-Witt theorem :

THEOREM 1.8.1 : Every Lie algebra is isomorphic to a subalgebra of some attached Lie algebra A^{-} of an associative algebra *A*.

Next we will discuss shortly about Lie algebras whose

elements are linear operators. Let L be any given linear space and $\mathbf{C}(L)$ the associative algebra consisting of all linear operators on L, with respect to the usual compositions of maps. We form the attached algebra $\mathbf{C}^{-}(L)$, whose any subalgebra is called **operator Lie algebra**. The elements of an operator Lie algebra are linear operators f acting on L, and the Lie product is defined by :

$$(\forall x \in L)\Big[\, [f_1, f_2](x) = (f_1 f_2)(x) - (f_2 f_1)(x) \,\Big] \quad (1.8.24)$$

The importance of operator Lie algebra is exhibited by the following theorem :

THEOREM 1.8.2 : Every Lie algebra is isomorphic to a Lie algebra of linear operators.

In the case of a finite dimensional Lie algebra we have :

THEOREM 1.8.3 : Every finite n-dimensional Lie algebra is isomorphic to a subalgebra of a Lie algebra consisting of $n \times n$ matrices, with the Lie product being the usual commutator of matrices.

DEFINITION 1.8.7 : A subalgebra D of a Lie algebra L is called **ideal** or **invariant subalgebras** if the following property is verified :

$$(\forall d\in D)(\forall x\in L)\Big[\ [d,x]\in D \wedge [x,d]\in D\ \Big] \qquad (1.8.25)$$

DEFINITION 1.8.8 : A Lie algebra L with dimension n>1, which has no other ideals than the trivial ones, i.e., $\{0\}$ and L, is called **simple**. If L has no, (nonzero), Abelian ideal, then L is said to be **semisimple**.

Let L_1 and L_2 be subalgebras of a Lie algebra L. We define :

$$[L_1,L_2] = \text{subalgebra spanned by the set } \{ [x_1,x_2] \text{ where } x_1\in L_1,\ x_2\in L_2\} \qquad (1.8.26)$$

So a subalgebra D of L is an ideal if $[L,D]\subset D$. We construct now the ideals :

$$L^{(1)}=[L,L]\ ,\ L^{(2)}=[L^{(1)},L^{(1)}]\ ,\ \dots\ ,\ L^{(k)}=[L^{(k-1)},L^{(k-1)}],\dots \qquad (1.8.27)$$

obtaining the series of derived ideals :

$$L\supset L^{(1)}\supset L^{(2)}\supset\dots\supset L^{(k)}\supset\dots \qquad (1.8.28)$$

DEFINITION I.8.9 : A Lie algebra L is called **solvable** if there exist a k∈ℕ such that $L^{(k)}=\{0\}$.

Every Abelian Lie algebra is solvable, since then $L^{(1)}=[L,L]=\{0\}$. On the ohter hand, a solvable Lie algebra with k>1 cannot be simple, nor even semisimple. Indeed, a solvable Lie algebra surely has an ideal $L^{(k-1)}$, and since

$[L^{(k-1)},L^{(k-1)}]=\{0\}$, this ideal is Abelian.

Another class of interesting ideals can be constructed in a similar manner :

$$L^2=[L,L]\ ,\ L^3=[L^2,L]\ ,\ \dots\ ,\ L^k=[L^{k-1},L]\ ,\ \dots \tag{I.8.29}$$

where

$$L\supset L^2\supset L^3\supset\dots L^k\supset\dots \tag{1.8.30}$$

DEFINITION 1.8.10 :If there exists a $k\in\mathbb{N}$ such that $L^k=\{0\}$, then the Lie algebra L is called **nilpotent**.

Every nilpotent Lie algebra is solvable, but the converse is not true. It then follows that a nilpotent Lie algebra cannot be simple (if k>2), nor semisimple.

In the sequel we mention some nonassociative algebras which, along with Lie algebras, are often met in physics.

DEFINITION 1.8.11 : **Jordan algebras** are nonassociative algebras J equipped with the product xy which obeys the following two axioms :

$$(\forall x,y\in J)\left[\ xy=yx\ \right] \tag{1.8.3I}$$

$$(\forall x,y\in J)\left[\ x^2(yx)\ =\ (x^2y)x\ \right] \tag{1.8.32}$$

If A is an associative algebra and if we define :

$$x y = \frac{1}{2}(xy+yx) \qquad (1.8.33)$$

we can easily see that we obtain a Jordan algebra, which will be denoted by A^+. In contrast to Lie algebras, however, we do not have an analog of the Poincaré-Bikhoff-Witt theorem : there exist Jordan algebras, which are not subalgebras of any A^+ algebra.

DEFINITION 1.8.12 : **Poisson algebras** are those associative algebras P, in which we introduce an additional, nonassociative internal composition law, called **Poisson bracket** and denoted by $[x,y]_p$. We set the following axioms (in addition to the already existing ones on $x+y$, αx, xy) :

$$(\forall x\in P)\Big[\ [x,x]_p=0\Big] \qquad (1.8.34)$$

$$(\forall x,y,z\in P)\Big[\ [x,(y+z)]_p=[x,y]_p+[x,z]_p \wedge [(x+y),z]_p=[x,z]_p+[y,z]_p\ \Big] \qquad (1.8.35)$$

$$(\forall x,y\in P)(\forall \alpha\in \mathbb{F})\Big[\ \alpha[x,y]_p=[\alpha x,y]_p=[x,\alpha y]_p\ \Big] \qquad (1.8.36)$$

$$(\forall x,y,z\in P)\Big[\ [x,yz]_p=[x,y]_p z+y[x,z]_p\ \Big] \qquad (1.8.37)$$

A familiar example of Poisson algebras comes from classical mechanics : Let

$$P = \left\{ f \;/\; f:\mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R} \text{ , f=infinitely differentiable real valyed functions} \right\} \tag{1.8.38}$$

The set P have the structure of an ordinary associative algebra by pointwise sum, scalar product, and product :

$$(\forall x,y\in\mathbb{R})(\forall f,g\in P)\Big[\ (f+g)(x,y)=F(x,y)+g(x,y)\ \Big] \tag{1.8.39}$$

$$(\forall x,y\in\mathbb{R})(\forall f\in P)(\forall o\in\mathbb{R})\Big[\ (of)(x,y)=of(x,y)\ \Big] \tag{1.8.40}$$

$$(\forall x,y\in\mathbb{R})(\forall f,g\in P)\Big[\ (fg)(x,y)=f(x,y)g(x,y)\ \Big] \tag{1.8.41}$$

Define now the Poisson bracket :

$$[f,g]_p = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} \tag{1.8.42}$$

It is easy to see that all axioms of a Poisson algebra are fulfilled as well as the Jacobi identity. Hence, the Poisson algebra of classical mechanics is also a Lie algebra, (but it is not a commutator algebra).

DEFINITION 1.8.13 : An algebra A is called **flexible** when

$$(\forall x,y\in A)\Big[\ [x,y,x] = (xy)x-x(yx) = 0\ \Big] \tag{1.8.43}$$

The flexibility axiom can be equivalently formulated (for fields of characteristic $p\neq 2$) in terms of the relation :

$$(\forall x,y,z\in A)\Big[\ [x,y,z] + [z,y,x] = 0\ \Big] \qquad (1.8.44)$$

DEFINITION I.8.14 : An algebra A (associative or not), is called **general Lie-admissible algebra** (Albert 1948a), if it is characterized by a product xy, such that the attached product $[x,y]_A=xy-yx$ is Lie. This implies the unique axiom :

$$(\forall x,y,z\in A)\Big[\ (x,y,z)+(y,z,x)+(z,x,y) = (z,y,x)+(y,x,z)+(x,z,y)\Big] \qquad (1.8.45)$$

where (x,y,z) is the associator of x,y,z.

Note that Lie algebras are a particular case of the Lie-admissible algebras. In fact, given an algebra L with product $xy=[x,y]_A$, the attached algebra L^- has the product $[x,y]_L=2[x,y]_A$ and thus L is Lie-admissible.

Therefore, the classification of the Lie-admissible algebras contains all Lie algebras. In particular Lie algebras enter in the Lie-admissible algebras in two-fold way : first, in their classification and second, as the attached antisymmetric algebras. Finally, associative algebras are trivially Lie-admissible.

The first abstract realization of the general Lie-admissible algebras was given by Santilli, ((1978b),

Sect. 4.14), and can be written :

$$(\forall x,y\in A)\left[\ (x,y)_A \equiv xry-ysx,\ r,s \text{ fixed } \in A,\ r\neq s\ r,s\neq 0\ \right] \tag{1.8.46}$$

The first realization of L in classical mechanics was also identified by Santilli (1969) and (1978a) and it is given by the following product for functions f(r,p) and g(r,p) in $T^*E(r,\delta,A)$

$$L\ :\ (f,g) = \frac{\partial f}{\partial r^k}\frac{\partial g}{\partial p_k} \tag{1.8.47}$$

namely, the general, nonassociative Lie-admissible algebras are at the foundations of the structure of the conventional Poisson brackets, which can be written :

$$[f,g]_{Poisson} = [f,g]_L = (f,g)-(g,f) \tag{1.8.48}$$

DEFINITION 1.8.15 : An algebra A is called **flexible Lie-admissible** if it is flexible and verifies the property :

$$(\forall x,y,z\in A)\left[\ (x,y,z)+(y,z,x)+(z,x,y)=0\ \right] \tag{1.8.49}$$

An abstract realization of the flexible Lie-admissible product (Santilli 1978b) is given by :

$$(\forall x,y,z\in A)\left[\ (x,y)_A = \lambda xy-\mu yx\ ,\ \lambda,\mu\in\mathbb{F}\ \right] \tag{1.8.50}$$

No classical realization of flexible Lie-admissible

algebras has been identified until now. As an example, the brackets on $T^*E(r,\delta,\mathbb{R})$

$$(f,g) = \lambda \frac{\partial f}{\partial r^k} \frac{\partial g}{\partial p_k} - \mu \frac{\partial g}{\partial r^k} \frac{\partial f}{\partial p_k} \tag{1.8.51}$$

are Lie-admissible, but violate the flexibility law.

DEFINITION 1.8.16 : An algebra A is called **general Jordan-admissible algebra** if it is characterized by a product xy, such that the attached symmetric product $\{x,y\}_A = xy+yx$ is Jordan, i.e. it verifies the axioms (1.8.31) and (1.8.32). The axiom (1.8.32) more generally can be written as :

$$(\forall x,y,z\in A)\left[\ (x^2,y,x)+(x,y,x^2)+(y,x^2,x)+(x,x^2,y) = 0\ \right] \tag{1.8.52}$$

Again, associative and Jordan algebras are trivially Jordan-admissible. Also, Jordan algebras enter in the Jordan-admissible algebras in a two-fold way, in the classification of the latter, as well as the attached symmetric algebras.

It is important for the operator formulation of the isotopies to point out that the Lie-admissible product (1.8.46) is jointly, Lie admissible and Jordan-admissible, (Santilli 1978b), because the attached symmetric product characterizes a special commutative Jordan algebra.

Finally, we should note that the classical product (1.8.47) is Lie- admissible and not jointly Jordan-

admissible.

DEFINITION 1.8.16 : An algebra A is called **flexible Jordan-admissible** (Albert 1948, Santilli 1978a,b), if it is characterized by a product xy such that the following axioms hold :

$$(\forall x,y\in A)\left[\ x(yx)=(xy)x\ \right] \qquad (1.8.53)$$

$$(\forall x,y\in A)\left[\ x^2(yx)+x^2(xy) = (x^2y)x+(x^2y)y\ \right] \qquad (1.8.54)$$

The flexible Lie-admissible product (1.8.50) is also flexible Jordan-admissible, but the classical product (1.8.51) is only Lie-admissible, and not flexible Lie-admissible nor Jordan-admissible.

DEFINITION 1.8.17 : An "**isoalgebra**" (Santilli 1978a), or simply an "**isotope**" $\hat{A}$ of an algebra A with elements x,y, z,... and product xy over a field $\mathbb{F}$, is the same vector space A but defined over the isofield $\hat{\mathbb{F}}$, equipped with a new product $x*y$, called "**isotopic product**", which is such to verify the original axioms of A.

Thus by definition, the isotopic lifting of an algebra does not alter the type of algebra considered.

Let us review the isotopies of the primary algebras listed above, beginning with the associative algebras (Santilli 1978a,b 1991a).

Given an associative algebra A with product xy over a field $\mathbb{F}$, its simplest possible isotope $\hat{A}$, hereon called **associative-isotopic** or **isoassociative algebra**, is given by

$$\hat{A}_1 : \ (\forall x, y \in A) \left[\ x*y \equiv \alpha xy \ , \ \alpha \in \mathbb{F} \text{ with } \alpha = \text{fixed and} \neq 0 \ \right] \tag{1.8.55}$$

and is called a **scalar isotopy**. The preservation of the original associativity is trivial in this case.

A second less trivial isotopy is the fundamental one of the Lie-Santilli theory, and it is characterized by the product :

$$\hat{A}_2 : \ (\forall x, y \in A) \left[\ x*y \equiv xTy \ \right] \tag{1.8.56}$$

where T is a nonsingular (invertible) and Hermitean (real valued and symmetric) element not necessary belonging to the original algebra A. The associativity of product (1.8.56) can also be readily proved.

Note the necessary condition, from Definition 1.8.17 that the isoproduct and isounit in $\hat{A}$ and $\hat{\mathbb{F}}$ coincide. That is, if in isofield $\hat{\mathbb{F}}$ the isoproduct is defined by $\alpha*\beta \equiv \alpha T \beta$ with isounit $\hat{I} = T^{-1}$, then the isoproduct in the isoalgebra $\hat{A}$ is defined by $x*y \equiv xTy$ with the same isounit $\hat{I} = T^{-1}$. This is the technical reason for the lifting of the universal enveloping associative algebras of a Lie algebra into a form whose center coincides with the isounit of the underlying isofield. Also the identity of the isoproduct and isounit

for A and $\mathbb{F}$ occurs in the associative cases (1.8.55) and (1.8.56), but does not hold in general, e.g., for nonassociative algebras. This is due to the lack of general admittance of a unit, while such a unit is always well defined in the underlying field.

A third significant isotopy of an associative algebra was given by Santilli, (1981), and it is characterized by the product :

$$\hat{A}_3 : (\forall x,y\in A)\left[\ x*y \equiv wxwyw \ /\ w^2=w\neq 0 \text{ and fixed}\ \right] \tag{1.8.57}$$

Additional isotopies are given by the combinations of the preceding ones, such as :

$$\hat{A}_4 : (\forall x,y\in A)\left[\ x*y \equiv wxwTwyw \ /\ w^2=w\neq 0 \text{ and fixed}\ \right] \tag{1.8.58}$$

and

$$\hat{A}_5 : (\forall x,y\in A)\left[\ x*y \equiv \alpha wxwTwyw \ /\ \alpha\in\mathbb{F},\ w^2=w,\ \alpha,w,T\neq 0\ \right] \tag{1.8.59}$$

It is believed that the above isotopies (of which only the first three are independent) exhaust all possible isotopies of an associative algebra (over a field of characteristics zero), although this property has not been rigorously proved until now.

The issue is not trivial, physically and mathematically. In fact, any new isotopy of an associative algebra

implies a potentially new mechanics, while having intriguing mathematical implications.

Another important point is that isotopy (1.8.56) is preferable than (1.8.57) because the former possesses a well defined isounit, while the latter does not admit a consistent isounit, thus creating a host of problems of physical consistency in its possible use for an operator theory.

We now pass to the study of the central notion introduced by Santilli (1978a), the isotopes $\hat{L}$ of a Lie algebra L with product xy over a fielf $\mathbb{F}$, here called **Lie- Santilli algebra**, which are the same vector space L but equipped with a product $x{*}y$ over the isofield $\hat{\mathbb{F}}$ which verifies the left and right scalar and distibutive laws (1.8.3) and (1.8.4) and the axioms :

$$(\forall x,y\in L)\left[\ x{*}y+y{*}x=0\ \right] \tag{1.8.60}$$

$$(\forall x,y,z\in L)\left[\ x{*}(y{*}z)+y{*}(z{*}x)+z{*}(x{*}y)\ =\ 0\ \right] \tag{1.8.61}$$

Namely, the abstract axioms of the Lie algebras remain the same by assumptions.

The simplest possible realization of the Lie-Santilli product is that attached to isotopes $\hat{A}_1$, (1.8.45) :

$$\hat{L}_1:\ (\forall x,y\in L)\left[\ [x,y]_{\hat{A}_1} = x{*}y-y{*}x\ =\ \alpha(xy-yx)= =\alpha[x,y]_A\ \ \alpha\in\mathbb{F},\ \alpha\neq 0\ \right] \tag{1.8.62}$$

and it is also called a **scalar Lie-isotopy**. It is generally the first lifting of Lie algebras one can encounter in the operator formulation of the theory.

The second independent realization of the Lie-Santilli algebras is that characterized by the isotopy $\hat{A}_2$:

$$\hat{L}_2:\ (\forall x,y\in L)\Big[\ [x,y]_{\hat{A}_2} = x*y-y*x = xTy-yTx\ \Big] \quad (1.8.63)$$

The third, independent isotopy is that attached to $\hat{A}_3$,

$$\hat{L}_3:\ (\forall x,y\in L)\Big[\ [x,y]_{\hat{A}_3} = x*y-y*x = wxwyw-wywxw \quad w^2=w\neq 0\ \Big] \quad (1.8.64)$$

A fourth isotopy is that attached to $\hat{A}_4$. i.e.

$$L_4:\ (\forall x,y\in L)\Big[\ [x,y]_{\hat{A}_4} = x*y-y*x = wxwTwyw-wywTwxw \quad w^2=w\neq 0,\ w,T\neq 0\ \Big] \quad (1.8.65)$$

A fifth and final, (abstract), isotopy is that characterized by $\hat{A}_5$, i.e.

$$L_5:\ (\forall x,y\in L)\Big[\ [x,y]_{\hat{A}_5} = o[x,y]_{\hat{A}_4}\ \Big] \quad (1.8.66)$$

Note that the Lie algebra attached to the general Lie-admissible product (1.8.46) are not conventional, but isotopic. In fact, we can write :

$$[x,y]_L = (x,y)_A-(y,x)_A = (xry-ysx)-(yrx-xsy) =$$

$$= x(r+s)y-y(r+s)x \; xTy-yTx = x*y-y*x$$

$$\text{where} \quad r\neq s \;, \; r,s,T\neq 0 \;, \quad T=r+s \qquad (1.8.67)$$

The following Proposition can be easily proved from properties of type (1.8.67).

PROPOSITION 1.8.1 : An abstract Lie-isotopic algebra $\hat{L}$ attached to a general, nonassociative, Lie-admissible algebra A, that is $\hat{L}\approx A^-$, can always be isomorphically rewritten as the algebra attached to an isoassociative algebra $\hat{A}$, that is $\hat{L}\approx\hat{A}^-$, and vice-versa, i.e.

$$\hat{L} \approx A^- \approx \hat{A}^- \qquad (1.8.68)$$

The obove property has the important consequence that *the construction of the abstract Lie-isotopic theory does not necessarily require a nonassociative enveloping algebra because it can always be done via the use of an isoassociative enveloping algebra*. In turn, this focuses again the importance of knowing all possible isotopes of an associative algebra, e.g., from the viewpoint of the representation theory.

CHAPTER II

LIE-SANTILLI ALGEBRAS

2.1 : STATEMENT OF THE PROBLEM

We are now sufficiently equipped to present the first main topics of this monograph, the isotopies of the primary structural theorems on Lie algebras, as derived in the original memoir (Santilli 1978a). In this chapter we shall essentially follows the presentation of Santilli (1982c), Charts 5.1-5.4, pages 154-183, and (1991a).

The term **"Lie theory"** is referred today to an articulated body of sophisticated mathematical tools encompassing several disciplines. Whether in functional analysis or in the theory of linear operators, the structure of the contemporary formulation of Lie's theory can be reduced to the following three parts :

$$
\begin{tikzcd}
 & \boxed{\text{Universal enveloping associative algebras } U} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\boxed{\text{Lie algebras } L} \arrow[rr, leftrightarrow] & & \boxed{\text{Lie groups } G}
\end{tikzcd}
$$

As duly emphasized in the mathematical literature, (Jacomson 1962, Dixmier 1977, and others), a truly fundamental part of Lie's theory is the **universal enveloping algebra** U, which is obtained from a tensor algebra by means of a quotient construction. This associative algebra, is important because it allows us to translate questions about Lie algebras into corresponding questions about associative algebras. We know that the Lie group theory reduces locally to the theory of Lie algebras. By means of the enveloping algebra, the theory of Lie algebra is reduced to the even better understood theory of associative algebras.

Intuitively, the universal enveloping associative algebra $U(L)$ constists of all polynomials in elements of the Lie algebra L and a unity element, with the Lie product $[x,y]$ of elements x and y in L being identified with the commutator $xy-yx$. More formally, we can construct the universal enveloping algebra as follows : Since the Lie algebra L is a vector space, it is possible to construct its contravariant tensor algebra $T(L)=\mathbb{F}\oplus L\oplus(L\otimes L)\oplus\ldots$ In this associative algebra $T(L)$, we consider the two-side ideal K generated by the set of all elements of the form :

$$[x,y]-(x\otimes y-y\otimes x)$$

where x and y are elements of L. The ideal K thus contains the differences between Lie algebra products and the corresponding commutators in the associative tensor algebra. If we consider the associative quotient algebra

$$U(L) = \frac{T(L)}{K} \qquad \text{(A)}$$

then Lie products will not be distinguished from commutators since they belong to the same coset. The associative algebra $U(L)$ is called **universal enveloping algebra** of the Lie algebra L. As with any associative algebra, we can also make $U(L)$ a Lie algebra using the commutator operation as the Lie product. If we do this, we can consider L to be injected homomorphically into $U(L)$, considered as a Lie algebra.

The associative algebra $U(L)$ is useful because of the following property : Suppose that A is an arbitrary associative algebra, and that A is also given the commutator Lie algebra structure. Any homomorphism of L into A, considered as a Lie algebra, has a unique extension to an associative algebra homomorphism of $U(L)$ into A. Now a representation of a Lie algebra L is a Lie homomorphism of L into the associative algebra of all linear operators on the module, with the Lie product of two linear operators being their commutator. Thus every representation of a Lie algebra L can be extented to a representation of its universal enveloping associative algebra $U(L)$, and we see that every module over L can also be regarded as a module over its enveloping algebra $U(L)$. This idea is central to certain proofs of complete reducibility for modules over semisimple Lie algebras which are based on the universal enveloping algebra.

In fact, the algebra U provides a symbiotic characte-

rization of both the Lie algebra and the Lie groups. This is due to the fact that the basis of U, (which is constructed via the Poincaré-Birkhoff-Witt Theorem, is given by an infinite number of suitable polynomial powers of the generators X_i of G such as :

$$U : 1 \in F; X_i; X_i X_j (i \leq j); X_i X_j X_k (i \leq j \leq k); \ldots \qquad (2.1.1)$$

where the product $X_i X_j$, etc., are associative. It then follows that the Lie algebra L :

$$L : [X_i, X_j] = X_i X_j - X_j X_i = C^k_{ij} X_k \qquad (2.1.2)$$

is, (homomorphic to), the attached algebra U^- of U. The Lie group G of L is then the infinite power series :

$$G : \exp(\theta k X_k) = 1 + \frac{\theta^k}{1!} X_k + \frac{\theta^i \theta^j}{2!} X_i X_j + \ldots \qquad (2.1.3)$$

which, evidently, can be properly defined and treated only in the enveloping algebra, (note that all terms from $X_i X_j$ on are outside the Lie algebra). One can then see why fundamental aspects of Lie algebras, (such as the representation theory), are treated by mathematicians within the context of its **enveloping algebra**.

On physical grounds, the role of the enveloping algebra is equally crucial, even though not sufficiently emphasized in the current literature. For instance, a frequent physical problem is the computation of the magnitude of physical quantities, such as the magnitude, (eigenvalue), of the angular momentum, (operator), $M=$

$||\mathbf{M}||^{1/2}$. While the components M_i of $\mathbf{M}$ are elements of the Lie algebra **SO(3)**, the quantity $\mathbf{M}^2$ is outside **SO(3)** and can only be defined in the, (center of), the enveloping algebra U(**SO(3)**). Thus, while the Lie algebra **SO(3)** essentially characterizes the components of the angular momentum and their commutation rules, the envelope U(**SO(3)**) characterizes 1) the components M_i; 2) their commutation relations via the attached rule $U^- \approx$ **SO(3)**; 3) the magnitude of the angular momentum $\mathbf{M}^2$; 4) the exponentiation to the Lie group of rotations; 5) the representation theory, etc. In short, we can state that a truly primitive part of the contemporary formulation of Lie's theory is its universal enveloping associative algebra.

Once the mathematical and physical motivations of this occurence are uderstood in full, the need for a suitable generalization of Lie' theory becomes unavoidable. Lie algebras emerge in Physics at the trully fundamental part, the brackets of the time evolution. The above remarks then imply that the primitive algebraic structure of the time evolution is the enveloping algebra. Santilli, (1978b) points out that the enveloping algebra of the time evolution of Hamilton Mechanics is *nonassociative*, by therefore being not directly compatible with the contemporary formulation of Lie's theory. In fact, he essentially indicated that the conventional Poisson brackets :

$$L : [X_i, X_j] = \frac{\partial X_i}{\partial a^\mu} \omega^{\mu\nu} \frac{\partial X_j}{\partial a^\nu} = \frac{\partial X_i}{\partial r^k} \frac{\partial X_j}{\partial p_k} - \frac{\partial X_j}{\partial r^k} \frac{\partial X_i}{\partial p_k} \tag{2.1.4}$$

are the attached brackets of the algebra :

$$\mathbf{U} : (X_i, X_j) = \frac{\partial X_i}{\partial r^k} \frac{\partial X_j}{\partial p_k} \tag{2.1.5}$$

which is nonassociative; that is, the vector space U of elements X_i and their polynomial powers, over the field $\mathbb{R}$ of real numbers equipped with product (2.1.5), is first of all an algebra since it verifies the left and right distributive laws and the scalar law. Secondly, this algebra turns out to be nonassociative because of properties :

$$((X_i, X_j), X_k) \neq (X_i, (X_j, X_k)) \tag{2.1.6}$$

Since associative and nonassociative algebras are different algebras, without a known interconnecting mapping, Santilli argues that the insistence on the associative character of the envelope would literally prohibit the conventional formulation of Hamiltonian Mechanics, that according to time evolution (2.1.4). He therefore advocates a dual generalization of Lie's Theory, (see the preceding paper, Santilli (1978b, pp 298-375) according to the following classification.

I. **Contemporary Formulation of Lie's Theory**. This is the formulation available in the contemporary literature, and it is expressed via an envelope with conventional associative product X_iX_j (e.g., the conventional product of matrices or operators).

II **Lie-Isotopic Generalization of Lie's Theory**. This is a first generalization based on envelopes which are still associative yet are formulated via the most general possible associative product, say, $X_i{*}X_j$, whose attached product $X_i{*}X_j-X_j{*}X_i$, is Lie.

III **Lie-Admissible Generalization of Lie Theory**. This is the largest possible generalization of Lie's theory conceivable at this time. It is based on envelopes that are Lie-admissible that is, on envelopes with the most general possible nonassociative product, say, (X_i,X_j), whose attached product $(X_i,X_j)-(X_j,X_i)$ is Lie.

We think that a few introductory remarks may help the reader to reach a better mathematical and physical understading of the generalization under considaration.

A difficulty generally experienced by mathematicians trying to see the need for a generalization of Lie's theory is that simple Lie algebras over a field of characteristic zero have been classified and are given by the well known

Cartan classificaton[1]. In fact, the Poincaré-Birkhoff-Witt theorem essentially ensures that all Lie algebras over a field of characteristic zero can be obtained as the attached algebras of enveloping algebras with the conventional associative product X_iX_j. Thus the classification of Lie algebras has been already achieved by Formulation I. The point is that *generalizations II and III are not intended for the classification. Instead, they are intended for the formulation of Lie's theory in the most general possible (rather than simplest possible) form, as a necessary condition for its direct applicability in physics.* Generalizations II and III are, of course, expected to recover Cartan classification. But this is a minor aspect of the issue. The issue is that of abandoning the conventional mathematical treatment of Lie algebra,

$$[X_i,X_j] = X_iX_j-X_jX_i \qquad (2.I.7)$$

(I) It is appropriate to recall here that the classification of Lie algebras over a field of characteristics $p\neq0$ is far from complete. The generalizations of Lie's theory here reffered to are intended primarily for the conventional case of characteristic zero which is the most important for current physical applications (in fact, no physical application is known at this time for algebras and/or fields of characteristic $p\neq0$).

where $X_i X_j$ is the conventional associative product, in favor of the most general conceivable product :

$$[X_i, X_j]^* = (X_i, X_j) - (X_j, X_i) \qquad (2.1.8)$$

where (X_i, X_j) is a nonassociative Lie-admissible product. Only in this way does the theory acquire a form suitable for direct application to machanics while possessing trivial realization (2.1.7) as a particular case. At any rate, while the formulation of Lie's theory for structure (2.1.8) includes that of structure (2.1.7) as a particular case, the opposite is not necessarily true[(2)]. As an example, the current formulation of the representation theory is inapplicable to Lie algebras (2.1.8) beginning from its foundations (necessary and sufficient conditions for a representation to be faithful, Ado's theorem, etc). At a deeper analysis, it soon emerges that the alteration of the associative character of the envelope into a nonassociative form demands the reformulation of the entire theory.

Perhaps an effective way for a mathematician to see the need of reformulating Lie's theory is through a comparative analysis with the corresponding situation in the symplectic and contact geometries, for which no reformula-

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(2) As will be soon evident, nonassociative products exist under which the enveloping algebra can be trivially reduced to an associative form.

tion is needed for local-differential systems. In essence, these geometries, in their most abstract and general form (the coordinate-free form), present a body of notions, properties and theorems hich preserve their validity under all possiblerealizations of the symplectic and contact forms. For instance, all the parts of the symplectic geometry dealing with symplectic two-forms :

$$\Omega_2 = d\theta \qquad (2.1.9)$$

preserve their validity regardless of where the two-form is the canonical form :

$$\omega_2 = dp_k \wedge dr^k \qquad (2.1.10)$$

or the most general possible Birkhoffian form (Santilli 1978a, 1982c)

$$\Omega_2 = \frac{1}{2}\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right) da^\mu \wedge da^\nu \quad a=(r,\ ) \qquad (2.1.11)$$

The crucial character of the theory, that of being applicable to all possible realizations, is lost for the contemporary formulation of Lie's theory. In fact, if the enveloping algebra is generalized from the trivial product $X_i X_j$ to a more general product $X_i * X_j$ (e.g., $X_i * X_j = X_i T X_j$, with T fixed and nonsingular, then the notion of Lie group (2.1.3) is generalized into structures, for instance, of the

type :

$$G^{*} : \exp(\theta^{k}X)|_{*} = \mathbf{1} + \frac{\theta}{1!}X^{k} + \frac{\theta^{i}\theta^{j}}{2!}X_{i}{*}X_{j} + \ldots$$

(2.1.12)

The fact that the notions, properties, and theorems developed for the conventional structure (2.1.3) are not necessarily applicable to the more general structure (2.1.12) is established, for instance, by the fact that $\mathbf{1}$ is no longer the unit of the envelope, trivially, because now $\mathbf{1}{*}X_{i} \neq X_{i} \neq X_{i}{*}\mathbf{1}$.

Remarkably, while the symplectic and contact geometries have been developed by keeeping the most general possible realizations of the two-forms in mind, the theory of Lie algebras has been developed until now for the simplest possible realization of the Lie algebra product. The Lie-isotopic generalization of Lie's theory is advocated here in order to recover the compatibility of the formulation with the symplectic and contact geometries, that is, to reach algebraic notions, properties, and theorems which are directly applicable to their most general possible realizations, in the same way as it occurs for the geometric counterparts. The Lie-admissible generalization of Lie's theory,instead, is intended as the algebraic counterpart of the symplectic-admissible generalization of the symplectic geometry (Santilli 1982d).

A further point which should be clarified is that the Lie-isotopic generalization of Lie's theory is not directly applicable to the Hamiltonian as well as the Birkhoffian Mechanics. In fact, the envelope is still associative by conception, while algebra (2.1.5) is already nonassociative for Hamiltonian mechanics, and this algebraic character clearly persists for the covering Birkhoffian Mechanics. The theory under consideration is merely an intermediate step prior to the full treatment of type III. Nevertheless, a possibility exists that the theory is applicable in a specific case, on account of the following property. Often, when structure (2.1.B) is worked out, it implies the possible reformulation :

$$[X_i, X_j]^* = (X_i X_j) - (X_j X_i) = X_i * X_j - X_j * X_i \qquad (I.2.13)$$

An example is given by the Lie-admissible product

$$(X_i, X_j) = X_i R X_j - X_j S X_i$$

with R and S fixed and nonsingular, and $X_i R$, RX_j, etc., associative. Then we have :

$$(X_i, X_j) - (X_j, X_i) = (X_i R X_j - X_j S X_i) - (X_j R X_i - X_i S X_j) =$$

$$= X_i T X_j - X_j T X_i = X_i * X_j - X_j * X_i \qquad T = R + S \qquad (2.I.14)$$

where $X_i T X_j$ is clearly isotopic associative. Thus, in certain instances, the intermediary Lie-isotopic generalization may be sufficient.

For the case of the Hamiltonian Mechanics, one can attempt modifications of product (2.1.5) into more general Lie-admissible forms of the type :

$$(X_i,X_j)^* = \frac{\partial X_i}{\partial r^k}\frac{\partial X_j}{\partial p_k} + \frac{\partial X_i}{\partial r^i}\alpha^{ij}\frac{\partial X_j}{\partial r^i} + \frac{\partial X_j}{\partial p_i}\beta_{ij}\frac{\partial X_j}{\partial p_j} \tag{2.1.15}$$

that is, modifications which are such as to preserve the conventional Poisson brackets as the attached Lie brackets. With the understanding that modifications (2.1.15) remain nonassociative in general, it may be that the associative law is regained in particular cases. (One can easily see that the associative law cannot in general be verified for product (2.1.15) because, for instance, the expression $((X_i,X_j),X_k)$ implies only first-order derivatives for X_k, while the expression $(X_i,(X_j,X_k))$ implies second-order derivatives for X_k. Nevertheless, restrictions on the functional depedence of the generators are conceivable under which $((X_i,X_j),X_k) = (X_i,(X_j,X_k))$).

The important point is that, even when the associative character of the envelope is regained via extensions of type (2.1.15), the enveloping algebra is not of the trivial type X_iX_j, but rather of the most general possible type $X_i{*}X_j$, As a result, assuming that the associative character of the envelope of classical mechanics is regained via (still unknown) methods, the isotopic generalization of

Lie's theory remains mandatory for its direct applicability. Lacking the generalization, one risks the application of existing theorems conceived for formulations I which are actually meaningless for physical models belonging to case II or III.

Some of the most remarkable and intriguing implications are those for particle physics. The only time evolution known at this point with a structure truly of type I (that is, with an associative envelope with trivial product $X_i X_j$) is that of **Heisenberg's** equations in quantum mechanics:

$$\dot{\tilde{A}} = \frac{1}{i}[\tilde{A},\tilde{H}] = \frac{1}{i}(\tilde{A}\tilde{H} - \tilde{H}\tilde{A}) \quad , \quad \tilde{A}\tilde{H} = \text{Associative product} \tag{2.1.16}$$

with fundamental brackets (in our unified notation $\tilde{a}=(\tilde{\mathbf{r}},\tilde{\mathbf{p}})$, of course, now reffered to as operators in a Hilbert space).

$$[\tilde{a}^\mu,\tilde{a}^\nu] = \tilde{a}^\mu\tilde{a}^\nu - \tilde{a}^\nu\tilde{a}^\mu = i\tilde{\omega}^{\mu\nu} = i\left(\overline{\left\|\frac{\partial R^0_\nu}{\partial a^\mu} - \frac{\partial R^0_\mu}{\partial a^\nu}\right\|^{-1}}\right)^{\mu\nu}$$

$$\tilde{R}^0 = (\tilde{\mathbf{p}},\tilde{\mathbf{0}}) \tag{2.1.17}$$

The mere identification of the possibilities of generalizing Lie' theory according to types II and III immediately implies the possibility of generalizing Heisenberg's equations accordingly.

In fact, Santilli (1978d, pp. 725 and 752) proposed the following Lie-isotopic generalization of Heisenberg's equations

$$\dot{\bar{A}} = \frac{1}{i}\,[\bar{A},\bar{H}]^{*} = \frac{1}{i}\,(\bar{A}\bar{T}\bar{H} - \bar{H}\bar{T}\bar{A}) \tag{2.1.18a}$$

$$[\tilde{a}_{\mu},\tilde{a}^{\nu}]^{*} = i\tilde{\Omega}^{\mu\nu}(\tilde{a}) = i\,\overline{\left(\left\|\frac{\partial R_{\nu}}{\partial a^{\mu}} - \frac{\partial R_{\mu}}{\partial a^{\nu}}\right\|^{-1}\right)} \tag{2.1.18b}$$

and the following Lie-admissible generalization

$$\dot{\bar{A}} = \frac{1}{i}\,[\bar{A},\bar{H}] = \frac{1}{i}\,(\bar{A}\bar{R}\bar{H} - \bar{H}\bar{S}\bar{A}) \tag{2.1.19a}$$

$$(\tilde{a}^{\mu},\tilde{a}^{\nu}) = iL^{\mu\nu}(\tilde{a}) = i(\tilde{\Omega}^{\mu\nu} + \tilde{T}^{\mu\nu}), \quad \tilde{T}^{\mu\nu} = \tilde{T}^{\nu\mu} \tag{2.1.19b}$$

As a matter of fact, generalizations of Lie's theory of types II and III were intended as mathematical tools for the proper treatment of the corresponding generalized equations of type (2.1.18) and (2.1.19). For these quantum mechanical applications the interested reader may inspect Santilli (1992a,b,c).

The generalizations of type II and III studied until now in these introductory words refer to the transition from conventional, linear, local-differential and canonical- Hamiltonian realizations of Lie's theory to covering formula-

tions which are nonlinear and noncanonical-nonhamiltonian , yet still local-differential. This is requested by the use of the conventional symplectic geometry, owing to its stricly local-differential topology.

The transition from the symplectic geometry to its isotopic covering, as presented later on in Chapter V, then permits the transition from local-differential-canonical realizations to their nonlinear-nonlocal-nonhamiltonian coverings.

2.2 : ISOTOPIC GENERALIZATION OF UNIVERSAL ENVELOPING ASSOCIATIVE ALGEBRAS.

In this section we shall first review the definition of universal enveloping associative algebra, and the methods for the construction of its basis according to the Poincaré-Birkhoff-Witt theorem, (see, e.g., Jacobson 1962). We shall then present their *isotopic* generalizations, that is, generalizations which preserve the associative character of the product. By keeping in mind the primitive character of the enveloping algebra in Lie's theory, the generalizations presented in this section render inevitable a corresponding reinspection of Lie algebras and of Lie groups initiated in the next section.

DEFINITION 2.I.I The universal enveloping associative algebra of a Lie algebra L is the set (U,τ) where U is an associative algebra and τ a homomorphism of L into the attached algebra U^- of U satisfying the following properties. If U' is another associative algebra and τ' a homomorphism of L into U', a unique homomorphism γ of U into U' exists such that $\tau'=\tau\gamma$; i.e., the following diagram is commutative

$$\begin{tikzcd} & U^- \arrow[dd, "\gamma"] \\ L \arrow[ur, "\tau"] \arrow[dr, "\tau'"'] & \\ & U'^- \end{tikzcd} \qquad (2.2.1)$$

Whenever an algebra U belongs to the context of the definition above, we shall write $U(L)$. All Lie algebras are assumed, for simplicity, to be finite-dimensional. Also, all algebras and fields are assumed to have characteristic zero, and the basis of all Lie algebras is ordered.

The construction of the enveloping algebra $U(L)$ is conducted as follows :

Consider the algebra L as a vector space with basis given by the (ordered set of) generators X_i, $i=1,\ldots,m$. The tensorial product $L\otimes L$ is the ordinary Kronecker (or direct) product of L with itself as a vector space. Such a tensorial product constitutes an algebra because it satisfies the distributive and scalar laws. Also, the algebra is associative because the Kronecker product is associative. The most general possible, associative, tensor algebra which can be constructed on L as vector space is given by :

$$T = \mathbf{F}I\oplus L\oplus(L\otimes L)\oplus(L\otimes L\otimes L)\oplus\ldots \qquad (2.2.2)$$

where $\mathbf{F}$ is the base field and $\oplus$ denotes the direct sum. Let K be the ideal generated by all elements of the form :

$$[X_i,X_j]-(X_i\otimes X_j-X_j\otimes X_i) \qquad (2.2.3)$$

where $[X_i,X_j]$ is the product of L. Then the universal enveloping algebra $U(L)$ of L is given (or, equivalently, can be defined) by the quotient :

$$U(L) = \frac{T}{K} \qquad (2.2.4)$$

It is possimple to prove that the algebra (2.2.4) satisfies all the conditions of Definition 2.2.1 (see, for instance, Jacobson 1962 loc. cit.).

Of the most importance for mathematical and physical considerations is the identification of the basis of $U(L)$. The quantities :

$$M_s = X_{i1} \otimes X_{i2} \otimes \ldots \otimes X_{is} \tag{2.2.5}$$

are called **standard (nonstandard) monomials** of order s depending on whether the ordering :

$$i_1 \le i_2 \le \ldots \le i_s \tag{2.2.6}$$

is verified (not verified). It is possible to prove that every element of $U(L)$ can be reduced to a linear combination of standard monomials and (cosets of) 1. This yields the following fundamental theorem on enveloping associative algebras (Jacobson, loc. cit.).

THEOREM 2.2.1 (Poincaré-Birkhoff-Witt Theorem). The cosets of 1 and the standard monomials form a basis of the universal enveloping associative algebra $U(L)$ of a Lie algebra L.

The associative envelope $U(L)$, as presented, is still abstract in the sense that the product of $U(L)$is the

tensorial product $X_i \otimes X_j$, while the product used in physics (e.g., quantum mechanics) applications is the conventional associative product $X_i X_j$. Consider then the algebra :

$$\mathbf{A}(L) = \mathbf{F}I \oplus \mathbf{A}^{(1)} \oplus \mathbf{A}^{(2)} \oplus \ldots$$

$$\mathbf{A}^{(s)} = \{X_{i1}, X_{i2}, \ldots, X_{is}\} \quad , \quad i_1 \leq i_2 \leq \ldots \leq i_s \qquad (2.2.7)$$

It is possible to prove that $U(L)$ is homomorphic to $\mathbf{A}(L)$, in line with Definition 2.2.1. Thus the algebra $\mathbf{A}(L)$ can be assumed as the universal enveloping associative algebra of L with basis :

$$\left\{ I,\ X_i,\ X_{i1}X_{i2},\ X_{i1}X_{i2}X_{i3}, \ldots \right\} \quad i_1 \leq i_2 \leq i_3 \ldots \qquad (2.2.8)$$

and arbitary elements : $X_{i1}^{k1} X_{i2}^{k2} \ldots X_{is}^{ks}$ (2.2.9)

where the X's are the generators of L. Notice that $\mathbf{A}(L)$ is infinite-dimensional. The **center** of $\mathbf{A}(L)$ is the set of all polynimials P(X) verifying the property :

$$\left[P(X), X_i\right]_A = 0 \qquad (2.2.10)$$

for all elements $X_i \in L$. Most important elements of the center are the so-called **Casimir invariants** of L.

We move now to the identification of the desired associative-isotopic generalization.

DEFINITION 2.2.2. (Santilli 1978b). The isotopically mapped universal enveloping associative algebra of a Lie algebra L is the set $\big((U,\tau),U^*,i,\tau^*\big)$ where

(i) (U,τ) is the universal enveloping associative algebra as per Definition 1;

(ii) i is an isotopic mapping of L, $i(L)=L^*$, as per Chart 4.2;

(iii) U^* is an another associative algebra generally nonisomorphic to U; and

(iv) τ^* is a homomorphism of L^* into U^{*-}, such that the following properties are verified.

If $U^{*\prime}$ is still another associative algebra and $\tau^{*\prime}$ a homomorphism of L^* into $U^{*\prime -}$, a unique homomorphism γ^* of U^* into $U^{*\prime}$ exists such that $\tau^{*\prime}=\gamma^*\tau^*$, and two unique isotopies $\hat{i}$ and $\hat{i}'$ exist for which $\hat{i}U=U^*$ and $\hat{i}'U'=U^{*\prime}$, i.e., the following diagram is commutative :

$$
\begin{tikzcd}
 & U^- \arrow[rr, "\hat{i}"] \arrow[dd, "\gamma"] & & U^* \arrow[dd, "\gamma^*"] \\
L \arrow[ur, "\tau"] \arrow[dr, "\tau'"'] \arrow[rr, "i"] & & L^* \arrow[ur, "\tau^*"] \arrow[dr, "\tau^{*\prime}"'] & \\
 & U'^- \arrow[rr] & & U^{*\prime -}
\end{tikzcd}
\qquad (2.2.11)
$$

Whenever an algebra U^* verifies the conditions of the definition above, we write $U^*(L)$. Again, for simplicity, we

assume that all Lie algebras are finite-dimensional, all algebras and fields have characteristic zero, and all Lie algebra bases are ordered.

We are now in a position to elaborate on the insufficiency of Definition 2.2.1, and the need of Definition 2.2.2 for the physical and mathematical studies under consideration in this monograph. We shall indicate first the mathematical aspect and then point out the physical profile.

The main idea of Definition 2.2.1 is, beginning with the basis of a Lie algebra L, to construct an enveloping algebra $U(L)$ such that $[U(L)]^{-} \approx L$. The more general idea of Definition 2.2.2 is, beginning also with the basis of a Lie algebra L, to construct an enveloping algebra $U^{*}(L)$ such that the attached algebra $[U^{*}(L)]^{-}$ is not, in general, isomorphic to L but rather is isomorphic to an isotope L^{*} of L, and we write(3)

$$[U^{*}(L)]^{-} \approx L^{*} \not\approx L \qquad (2.2.12)$$

The lack of unique association of a given basis with the envelope then ensures freedom in the realization of the associative product. Equivalently, we can say that within the context of Definition 2.2.1, a given basis essentially

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(3) Note that the scripture $U^{*}(L)$ (rather than $U^{*}(L^{*})$) is intended to stress precisely properties 2.2.12.

yields a single unique enveloping algebra and thus a single unique attached Lie algebra. On the contrary, within the context of Definition 2.2.2, a given an n-dimensionalbasis yields all possible enveloping algebras, and, thus all possible n-dimensionalLie algebras. Still equivalently, we can say that, as is conventional in the contemporary formulation of Lie's theory, nonisomrphicLie algebras are expressed via the use of different generators and the same realization of the Lie product. On the contrary, within the context of the isotopic formulation of Lie's theory, nonisomorphic Lie algebras can be obtained via the use of the same basis and different realizations of the Lie product. We can therefore state that all possible enveloping associative algebras can indeed be introduced according to Definition 2.2.1, which is therefore suitable for the classification of Lie algebras. Definition 2.2.2 is more general inasmuch as, besides permitting the introduction of all possible enveloping algebras, it also permits us to construct nonisomorphic algebras via the same basis, by therefore rendering necessary the use of the most general possible realizations of the associative product.

On physical grounds, these mathematical mechanisms are at the foundation of the Lie isotopic generalization of Hamilton's and Heisenberg's equations for closed non-self-adjoint interactions. As now familiar, the definition of physical quantities is independent of whether or not the system possesses nonpotential interactions. When these

interactions are admitted by the theory, they are represented via an alteraction of the Lie algebra product. As a result, when the Hamiltonian description of a closed self-adjoint system

$$\dot{A}(a) = [A, E_{tot}] = \frac{\partial A}{\partial a^{\mu}} \omega^{\mu\nu} \frac{\partial E_{tot}}{\partial a^{\nu}} \qquad a=(2,p) \qquad (2.2.13)$$

is generalized into a Birkhoffian form to represent the additional presence of internal, contact, nonpotential, interactions :

$$\dot{A}(a) = [A, E_{tot}]^{*} = \frac{\partial A}{\partial a^{\mu}} \Omega^{\mu\nu}(a) \frac{\partial E_{tot}}{\partial a^{\nu}} \qquad (2.2.14)$$

the basis of the original Lie algebra remains unchanged, together with the underlying carrier space ($\mathbb{R} \times T^{*}M$) and the field, and only the realization of the Lie algebra product (that is, the realization of the envelope) is permitted to change. As a result, the original Lie algebra L with basis $X_i(a)$ over $T^{*}M$ equipped with conventional Poisson brackets is mapped into the isotope L^{*}, which preserves the original basis $X_i(a)$ in the same local coordinates of $T^{*}M$, although it is now equipped with the generalized Poisson brackets, i.e.,

$$L : [X_i, X_j]_{(a)} = (X_i, X_j)_{(a)} - (X_j, X_i)_{(a)} \longrightarrow$$

$$L^{*} : [X_i, X_j]^{*}_{(a)} = (X_i, X_j)^{*}_{(a)} - (X_j, X_i)^{*}_{(a)} \qquad (2.2.15)$$

In the transition to the case of Heisenberg's equations, the situation is essentially the same and actually turns out to be more directly compatible with Definition 2.2.2. In fact, for consistency of the theory with its classical image, during the generalization of Heisenberg's equations :

$$\dot{U}(\bar{a}) = \frac{1}{i} [A, H] \qquad (2.2.16)$$

into the Lie-isotopic form :

$$\dot{U}(\bar{a}) = \frac{1}{i} [A, H]^* \qquad (2.2.17)$$

the nonpotential forces due to charge overlapping are expressed via the Lie-isotopic generalization of the product

$$L : [X_i, X_j] = X_i X_j - X_j X_i \longrightarrow L^* : [X_i, X_j]^* =$$

$$= X_i T X_j - X_j T X_i \qquad (2.2.18)$$

Mechanism (2.2.18) is clearly along Definition (2.2.2) rather than (2.2.1).

The alterative approach would be that of preserving the original simplest possible product and changing the basis in order to reach direct compatibility with Definition 2.2.1. However, this approach has a number of problematic aspects. First of all, it is centered on the loss of the direct physical meaning of the generators (e.g., the physical linear momentum in one dimansion, $p = m\dot{r}$, is replaced by abstract objects of the type $p = \alpha\exp(\beta r\dot{r})$. Secondly, the

approach does not permit the achievement of the direct universality, for all possible nonlinear, nonlocal and nonpotential systems. The removal of unnecessary restriction on the realization of the enveloping algebras is clearly preferable, both mathematically and physically.

Owing to the relevanve of mechanisms (2.2.15) and (2.2.18) for this monograph, it is important to give an exlicit example. To stress the fact that the ideas are not necessarily restricted to nonpotential interactions, we select an example of isotopy for the harmonic oscillator in a three-dimension Euclidean space.

We know that the nonisomorphic groups SO(3) and SO(2.1) are isotopic symmetries with respect to the Hamiltonians :

$$H(a) = \frac{1}{2}(p_x^2+p_y^2+p_z^2)+\frac{1}{2}(x^2+y^2+z^2) \qquad a=(r,p) \quad m=k=1 \tag{2.2.19a}$$

$$H^*(a) = \frac{1}{2}(p_x^2-p_y^2+p_z^2)+\frac{1}{2}(x^2-y^2+z^2) \tag{2.2.19b}$$

that is, they are symmetries leading to the same conservation laws of the components M_b, b=x,y,z, of the angular momentum via the use of Noether's theorem. Let us review the case again and reinterpret it in light of Definition 2.2.1 and 2.2.2.

The Hamiltonian realization of the symmetry SO(3) of H(a) is based on the Lie algebra of conserved quantities :

$$SO(3) \ : \ [M_x,M_y] = M_z \ , \ [M_y,M_z] = M_x \ , \ [M_z,M_x] = M_y \tag{2.2.20}$$

which is defined in terms of the conventional Poisson brackets

$$[M_b,M_c] = (M_b,M_c)-(M_c,M_b) \tag{2.2.21a}$$

$$(M_b,M_c) = \frac{\partial M_b}{\partial r^i}\delta^i_j\frac{\partial M_c}{\partial p_j} \quad ; \quad (\delta^i_j) = \begin{pmatrix} +1 & & 0 \\ & +1 & \\ 0 & & +1 \end{pmatrix} \tag{2.2.21b}$$

In the transition to the equaivalent Hamiltonian $H^*(a)$, the conserved quantities $\dot{M}_b$ clearly remain conserved, but the SO(3) symmetry is broken and is replaced by the nonisomorphic symmetry SO(2.1). The problem now is the construction of a realization of the SO(2.1) algebra (the Lorentz algebra in (2+1)-dimensions) whose generators are those of the nonisomorphic SO(3) algebra (the rotational algebra in three-dimensions). This can clearly be achieved if and only if one alters the Lie algebra product. An explicit realization has been identified by Santilli (1978a) and is given by the well-known commutation rules :

$$SO(2.1)\ :\ [M_x,M_y]^* = M_z\ ,\ [M_y,M_z]^* = -M_x\ ,\quad [M_z,M_x]^* = M_y$$

which are now expressed in terms of the generalized Poisson

(Birkhoffian) brackets.

$$[M_b,M_c]^* = (M_b,M_c)^*-(M_c,M_b)^* \qquad (2.2.23a)$$

$$(M_b,M_c)^* = \frac{\partial M_b}{\partial r^i} \omega^i_j \frac{\partial M_c}{\partial p_j} \; ; \; (\omega^i_j) = \begin{pmatrix} +1 & & 0 \\ & -1 & \\ 0 & & +1 \end{pmatrix} \qquad (2.2.23b)$$

Note that the insistence in the preservation of the same realization of the Lie algebra product, in this case, would prohibit the representation of the conservation of the angular momentum via a symmetry of the Hamiltonian $H^*(a)$.

The example considered therefore establishes that one given basis (the componets of the angular momentum $\mathbf{M}=\mathbf{r}\times\mathbf{p}$, $\mathbf{p}=m\dot{\mathbf{r}}$) can define a hierarhy of enveloping algebras and attached Lie algebras, depending on the selected realizations of the products, which is fully in line with diagram (2.2.11) and Definition 2.2.2. The example actually establishes not only the insufficiency of Definition 2.2.1 but also that of Definition 2.2.2 itself. In fact, the algebras (M_b,M_c) and $(M_b,M_c)^*$ are nonassociative, therefore demanding further generalization of Definition 2.2.1 for nonassociative enveloping algebras of type III even though the existence of a realization within the context of the Lie-isotopic generalization is expected to exist. For studies on the nonassociative, Lie-admissible generalization of

enveloping algebras one may consult Santilli (1978a) and Ktorides et al (1980).

Stated in different terms, the example establishes the generalizations of the conventional definition of the envelope of the Lie algebra of the group of rotations as per diagram (1)

$$\begin{tikzcd} & U^- \arrow[dd, "Y"] \\ SO(3) \arrow[ur, "T"] \arrow[dr, "T'"'] & \\ & U'^- \end{tikzcd} \tag{2.2.24}$$

into the Lie-isotopic form as per diagram (2.2.11).

$$\begin{tikzcd} & U^- \arrow[dd, "Y"] \arrow[rrr, "\hat{i}"] & & & U^* \arrow[dd, "Y^*"] \\ SO(3) \arrow[ur, "T"] \arrow[dr, "T'"'] \arrow[rr, "i"] & & SO(2.1) \arrow[urr, "T^*"] \arrow[drr, "T^{*\prime}"'] & & \\ & U'^- \arrow[rrr] & & & U^{*\prime -} \end{tikzcd} \tag{2.2.25}$$

which is expected for operator-type realizations (2.2.18). In addition, the example establishes that generalization (2.2.25) is only an intermediate step, prior to a more general nonassociative realization which is not considered here for the sake of brevity.

With a clear understanding of the new capabilities, as well as limitations, of the Lie-isotopic generalization, we pass now to the study of the generalization of Theorem

2.2.1.

The construction of an isotope $U^*(L)$ of $U(L)$ can be conducted as follows (Santilli 1978a, 1982c). Perform an isotopic mapping of the tensorial product $X_i \otimes X_j$ of $U(L)$

$$X_i \otimes X_j \longrightarrow X_i \imath X_j \qquad (2.2.26)$$

that is, any invertible modification of the product $\otimes$ via elements of $U(L)$, of the base manifold, and of the field, which preserves the distributive and scalar laws (to qualify as an algebra), as well as the associativity of the product (to qualify as an isotopy), i.e.,

$$(X_i \imath X_j) \imath X_k = X_i \imath (X_j \imath X_k) \qquad (2.2.27)$$

The product of two elements $X_i \imath X_j$ and $X_r \imath X_s$ is then given by

$$(X_i \imath X_j) \imath (X_r \imath X_s) = X_i \imath X_j \imath X_r \imath X_s \qquad (2.2.28)$$

and no ordering ambiguity arises because of the preservation of the associative character of the original product.Note that, for the more general nonassociative Lie- admissible generalization, the left- and right-hand sides of quantities (2.2.27) would be different. In this case all possible different orderings of the product must be taken into account

The isotope of the associative tensorial algebra T can then be written :

$$\mathrm{T}^* = \mathbb{F} \imath \otimes L \otimes (L \imath L) \otimes (L \imath L \imath L) \otimes \ldots \qquad (2.2.29)$$

Let $\mathbf{R}^*$ be the ideal of $\hat{\mathbf{T}}$ generated by :

$$[X_i,X_j]^* - (X_i \hat{\imath} X_j - X_j \hat{\imath} X_i) \tag{2.2.30}$$

where $[X_i,X_j]^*$ is the product in L^*. An isotopically mapped universal enveloping associative algebra $U^*(L)$ of the Lie algebra L can then be written :

$$U^*(L) = \frac{\hat{\mathbf{T}}}{\mathbf{R}^*} \tag{2.2.31}$$

Structure (2.2.31) is, by construction, the universal enveloping algebra of L^*, where L^* is realized via an isotopic mapping of L.

The remaining aspects of the theory of $\mathbf{U}^*(L)$ are essentially given by an isotopic mapping of the corresponding steps for $U(L)$ outlined above. The quantities

$$M^*_s = X_{i1} \hat{\imath} X_{i2} \hat{\imath} \ldots \hat{\imath} X_{is} \tag{2.2.32}$$

are called **isotopically mapped standard (nonstandard) monomials** depending on whether the following ordering condition :

$$i_1 \le i_2 \le \ldots \le i_s \tag{2.2.33}$$

is verified (not verified). In the reduction of an arbitrary element of $U^*(L)$

$$X^{k1}_{i1} \hat{\imath} X^{k2}_{i2} \hat{\imath} \ldots \hat{\imath} X^{kr}_{ir} \tag{2.2.34}$$

to standard monomial, a new feature arises, due to the fact

that the emerging combinations of these latter monomials may occur via functions on the base manifold. This, in turn, is because the isotopy $\otimes \longrightarrow I$ can be realized via functions of this type. We call these combinations F^*-linear, to differentiate them from the F-linear combinations for the conventional case, that is, combinations only via elements of the field. As we shall see in the next chart, these F^*-linear combinations have a precise interpretation within the context of the isotopic Lie's theory. Despite this generalization, the construction of the basis of $U^*(L)$ paralles that for $U(L)$ because $U^*(L)$ is a conventional envelope for L^*. The (inverse) isotopy then simply reduces L^* to L.

Finally, introduce the **isounit** I^* of $U^*(L)$, i.e., the quantity I^* such that $I^* I X_i = X_i I I^* = X_i$, $i=1,2,\ldots,n$, and the lifting of the original field $\mathbb{F}$ into the isofield $\mathbb{F}^*$ with **isounit** I^* (Sect. 1.4). Then the primary result of this section can be expressed via the following.

THEOREM 2.2.2 (Isotopic Generalization of the Poincaré-Birkhoff-Witt Theorem, Santilli, loc. cit.). The cosets of $I^* \in \mathbb{F}^*$ and the standard isotopically mapped monomials from a basis of the isotopically mapped universal enveloping associative algebra $U^*(L)$ of a Lie algebra L with basis :

$$I^*,\ X_i,\ X_{i1} I X_{i2},\ X_{i1} I X_{i2} I X_{i3} \qquad (2.2.35)$$

$$i_1 \leq i_2 \quad , \quad i_1 \leq i_2 \leq i_3$$

The distinction between the tensorial realization and

that used in practical applications is now lost. Indeed, the mapping $X_i \circledast X_j \longrightarrow X_i X_j$ can be considered, in the final analysis, a particular form of isotopy.

The explicit form of the basis depends on the assumed type of isotopy $\circledast \longrightarrow \imath$. In turn, this depends on the realization of the basis X_i of L, whether via matrices, quantum mechanical operators, or classical functions on phase space, etc.

Suppose that the X's are realized via matrices. Then an isotopy is provided by Equation (2.2.18). Let T be a polynimial on the X's (not necessarily on the center of $U^*(L)$)[4]. Then the explicit form of basis (2.2.35) is given by

$$\imath^*, \; X_i, \; X_{i1} T X_{i2}, \; X_{i1} T X_{i2} T X_{i3} \qquad (2.2.36)$$

$$i_1 \leq i_2, \quad i_1 \leq i_2 \leq i_3 \quad T = \text{fixed}, \quad {}^*\imath = T^{-1}$$

Needless to say, the isotopy $X_i X_j \longrightarrow X_i T X_j$ is only one example of possible associativity-preserving modifications of the product, and numerous additional forms exist. For

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(4) In a number of applications, the element T cannot actually be expressed via F^*-linear combinations of polynomials of the original basis, and as such, it is outside the original envelope.

instance, if W is an idempotent matrix ($W^2 = W$), then another associative isotopy is given by[5] :

$$X_i I X_j = WX_i WX_j W \tag{2.2.37}$$

A few comments in regard to the unit are now in order. As one can see, Theorem 2.2.2 has been formulated in its most general possible isotopic form, that with respect to an isofield $\mathbb{F}^*$ with isounit I^*, because needed for future aspects, such as the identification of the most general possible isotopies of the Lie algebras and groups.

The reader should be aware that the isotopy of the base field necessarily implies that of the underlying manifolds. Thus, Theorem 2.2.2 is formulated, strictly speaking, in a form compatible with the notion of isomanifolds of the next chapters, which is the main line of inquiry of this monograph.

Theorem 2.2.2 also admits a sort of "intermediary" formulation, that on ordinary manifolds, in which case the base field is conventional and the unit is the trivial one. In this latter case, Theorem 2.2.2 still holds with the conventional unit I replacing the isotopic one I^* in the infinite-dimensional basis (2.3.35). In fact, the latter

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(5) Intriguingly, isotopy (2.2.37) was introduced within the context of the studies for a possible isotopic generalization of Heisenberg's indeterminancy principle for strong interactions (Santilli 1981a, 1992c)

formulation was used in Santilli (1978a, 1982c), while the former is used in Santilli (1991a).

The origin of the above alternatives is related to the underlying geometry. In fact, if one selects the canonical realization of the symplectic geometry on T_1M over $\mathbb{F}$ then the conventional Theorem 2.2.1 applies. If one selects the Birkhoffian realization of the symplectic geometry also on T_1M over $\mathbb{F}$, then the "intermediate" version of Theorem 2.2.2 holds with the conventional unit $I \in \mathbb{F}$. On the contrary, if one selects the isosymplectic geometry on an isomanifold over $\mathbb{F}^*$, then the most general possible formulation of Theorem 2.2.2 holds.

The above three alternatives will be clearer in the next section when studing the corresponding construction of the Lie groups.

In summary the mathematical aspect here is that *the knowledge of a given set of generators deos not uniquely characterize a Lie algebra* because of the freedom in the selection of the enveloping algebra. The physical aspect treated is that *the knowledge of a Hamiltonian does not uniquely characterize the physical system* because such a characterization also depends on the explicit form of the brackets of the time evolution.

2.3 ISOTOPIC GENERALIZATION OF LIE'S FIRST, SECOND, AND THIRD THEOREMS

As well-known, an effective historical, and technical way of presenting Lie groups and Lie algebras is according to their original derivation by Sophus Lie via celebrated First,Second, and Third Theorems (Lie 1893). We shall first review these theorems and then show that they admit a consistent Lie isotopic, generalization which is compatible with the isotopic generalization of the enveloping algebra of the preceding section (Santilli 1978a, 1982b, 1991a). More specifically, the objective of this section is to show that the notion of connected Lie transformation group admits a generalization such that, when reduced in the neighborhood of the identity, admits Lie algebras in their most general possible realizations of the product.

From a physical viewpoint, there layers of the theory can be identified. First, we have the conventional formulation, and canonical-Hamiltonian transformations on a manifold, say, the phase space T^*M. In this case the conventional Lie's Theorems apply.

Second, we have transformations on an ordinary manifold which, besides preserving the local-differential character , are nevertheless noncanonical-Birkhoffian. In this case a first, "intermediate" isotopic form of Lie's Theorems apply, that characterizing the most general possible Lie

product although on a conventional manifold.

Finally, we have the most general possible transformations on an isomanifold, isotransformations, which are generally nonlocal-integral. In this latter case, we have the most general possible isotopies of Lie algebras and groups on isomanifolds.

This second is devoted to the "intermediate" presentation of the isotopies of Lie' Theorems, that on conventional manifolds, while the more general forms on isomanifolds will be considered later on in this analysis.

As clearly stated in Santilli (1978a, 1982c), a mathematical and physical motivation for the "intermediate" formulation is the following : Recall that the canonical realization of Lie's Theorems provides the algebraic counterpart of the symplectic geometry in canonical realization, i.e., the nowhere degenerate, exact and canonical two-form (2.1.10). These methods, in turn, provide the algebraic and geometric characterization of Hamiltonian systems.

The analysis of this section provides the algebraic counterpart of two-forms (2.1.11) which are the most general possible nowhere degenerate, exact and symplectic forms in local coordinates, called Birkhoffian two-forms by Santilli (loc. cit.). Still in turn, the transition from the Hamiltonian to the Birkhoffian realization of Lie's theory and of the symplectic geometry permits the representation of the most general possible nonlinear and nonpotential (varia-

tionally nonselfadjoint) systems although in their local-differential forms.

The subsequent third level of study, conducted later on in this monograph, provides the algebraic characterization of the most general possible isosymplectic two-forms, thus permitting the treatment of the most general possible nonlinear and non-Hamiltonian systems, this time in their nonlocal-integral form.

The emerging isotopic generalization of Lie's theory (that is, of the enveloping algebra, Lie algebras, Lie groups, representation theory) was used for the construction of the isotopic generalization of Galilei's and Einstein's relativities for closed non-self-adjoint systems (Santilli 1982c, 1991d, 1992b). Since the theory also admits operator-type realizations, its abstract formulation is expected to permit the joint treatment of closed, classical and quantum mechanical, nonpotential interactions, in much of the same way as the conventional abstract formulation of Lie's theory permits a joint treatment of closed classical and quantum mechanical interactions of potential/Hamiltonian type. The underlying physical objective is therefore to achieve, in due time, the generalization of the contemporary notion of interactions, with corresponding generalization of relativities and physical laws.

DEFINITION 2.3.1 Let M be a Hausdorff, second-countable, analytic, N-dimensional manifold with local coordinates a^{μ},

$\mu=1,2,\ldots,N$ (e.g., $M=T^*M$ or $\mathbb{R}\times T^*M$). The set of transformations on M depending on r-independent parameters θ^i, $i=1,2,\ldots,r$

$$a \longrightarrow a'=f(a;\theta)-\{f^\mu(a^\alpha;\theta^i)\} \quad (2.3.1)$$

is called a **Lie transformation group** when the following conditions are verified.

1. All functions f^μ are analytic in their variables.
2. For any given two transformations

$$a'=f(a;\theta)\ ,\ a^\mu=f(a';\theta') \quad (2.3.2)$$

a set of parameters exists

$$\theta''^i=g^i(\theta,\theta') \quad (2.3.3)$$

characterized by analytic functions g^i called **group composition laws**, such that :

$$a'' = f(a;\theta'') \quad (2.3.4)$$

3. Transformations (2.3.1) recover the identity transformation at the null value of the parameters, i.e.,

$$a=f(a;0) \quad (2.3.5)$$

4. Corresponding to each transformation (2.3.1), there is a unique inverse transformation

$$a=f(a';\theta^{-1}) \quad (2.3.6)$$

and thus the transformations are regular.
5. The combination of any transformation (2.3.1) with its inverse (2.3.6) yields the identity transformation.

The number r of independent parameters is called the **dimension** of the Lie group.

A central property of Lie transformation group is that they are *connected*; that is, they can be continuously connected to the identity. The primary idea of Lie's theorems is that, under the conditions indicated, the groups can be studied via their infinitesimal transformations, because a finite transformation can be recovered via infinite successions of infinitesimal transformations.

We shall review these ideas by following as closely as possible their original derivation. Consider transformations (2.3.1) and their identity

$$a' = f(a;\theta) \quad , \quad a = f(a;0) \tag{2.3.7}$$

and perform the infinitesimal variations

$$a' = a + da = f(a;\theta + d\theta) \ ; \ a + \delta a = f(a;\delta\theta) \tag{2.3.8}$$

where $d\theta$ and $\delta\theta$ represent two independent variations of the parameters. We can then write

$$da = \frac{\partial f(a;\theta)}{\partial \theta} \, d\theta \tag{2.3.9a}$$

$$\delta a = \left[\frac{\partial f(a;\theta)}{\partial \theta} \right]_{\theta=0} \delta\theta \tag{2.3.9b}$$

The transformation $\theta + d\theta$ can be interpreted as the product of

transformations relative to θ and $\delta\theta$, i.e.,

$$\theta^i+d\theta^i=\varphi^i(\theta,\delta\theta) \tag{2.3.10}$$

for which

$$\theta^i+d\theta^i=\varphi^i(\theta,0)+\left[\frac{\partial\varphi^i(\theta,o)}{\partial o^j}\right]_{\alpha=0}\delta\theta^j+\ldots \tag{2.3.11}$$

Thus we can write

$$d\theta^i=\mu^i_j(\theta)\delta\theta^j \quad , \quad \mu^i_j=\left[\frac{\partial\varphi^i(\theta,o)}{\partial\alpha^j}\right]_{\alpha=0} \tag{2.3.12}$$

The formula above represents a relation between $d\theta$ and $\delta\theta$ which can also be written

$$\delta\theta^j=\lambda^j_i(\theta)\delta\theta^i \quad , \quad \lambda^j_k\mu^k_i=\mu^k_i\lambda^j_k=\delta^j_i \tag{2.3.13}$$

By putting

$$u^\mu_i(a)=\left[\frac{\partial f^\mu(a;\theta)}{\partial\theta^i}\right] \tag{2.3.14}$$

and by using Equation (4b) can be written

$$da^\mu=u^\mu_k(a)\lambda^k_j(\theta)d\theta^j \tag{2.3.15}$$

In this way we reach **Lie's first theorem.**

THEOREM 2.3.1 When transformations (2.3.1) form a connected, m-dimensional, Lie group, then

$$\frac{\partial a^\mu}{\partial \theta^j} = u^\mu_k(a)\lambda^k_j(\theta) \qquad (2.3.16)$$

where the functions u^μ_k are analytic.

Let $A(a)$ be an (analytic) function of the variables. The infinitesimal Lie transformation $a \longrightarrow a+da$ induces a variation of $A(a)$ which can be written

$$dA = \frac{\partial A}{\partial a^\mu} u^\mu_j \delta\theta^j = \delta\theta^k \theta^\mu_k \frac{\partial A}{\partial \alpha^\mu} = \delta\theta^k X_k A \qquad (2.3.17)$$

The m-independent quantities :

$$X_k = X_k(a) = u^\mu_k(a)\frac{\partial}{\partial a^\mu} = \left[\frac{\partial f^\mu(a;\theta)}{\partial \theta^k}\right]_{\theta=0} \frac{\partial}{\partial a^\mu} \qquad (2.3.18)$$

are called the **infinitesimal generators** of the transformations (or of the group). For our later needs, we refer to the X's defined by Equations (2.3.18) as the **standard generarors.**

We are now interested in the (necessary and sufficient) conditions for transformations (2.3.1) to costitute a Lie group. By using the converse of the Poincaré lemma, they can be written :

$$\frac{\partial^2 a'^{\mu}}{\partial\theta^i\partial\theta^j} = \frac{\partial^2 a^{\mu}}{\partial\theta^j\partial\theta^i} \qquad (2.3.19)$$

that is

$$\frac{\partial u^{\mu}_k}{\partial\theta^i}\lambda^k_j + u^{\mu}_k\frac{\partial\lambda^k_j}{\partial\theta^i} = \frac{\partial u^{\mu}_k}{\partial\theta^j}\lambda^k_i + u^{\mu}_k\frac{\partial\lambda^k_i}{\partial\theta^j} \qquad (2.3.20)$$

Thus

$$u^{\mu}_k\left(\frac{\partial\lambda^k_j}{\partial\theta^i} - \frac{\partial\lambda^k_i}{\partial\theta^j}\right) = \lambda^k_j\frac{\partial u^{\mu}_k}{\partial\theta^i} - \lambda^k_i\frac{\partial u^{\mu}_k}{\partial\theta^j} =$$

$$= \lambda^k_j\frac{\partial u^{\mu}_k}{\partial a^{\nu}}\frac{\partial a^{\nu}}{\partial\theta^i} - \lambda^k_i\frac{\partial u^{\mu}_k}{\partial a^{\nu}}\frac{\partial a^{\nu}}{\partial\theta^j} =$$

$$= \lambda^r_j u^{\nu}_l \lambda^l_i\frac{\partial u^{\mu}_r}{\partial a^{\nu}} - \lambda^k_i u^{\nu}_l \lambda^l_j\frac{\partial u^{\mu}_r}{\partial a^{\nu}} \qquad (2.3.21)$$

Therefore,

$$u^{\nu}_i\frac{\partial u^{\mu}_j}{\partial a^{\nu}} - u^{\nu}_j\frac{\partial u^{\mu}_i}{\partial a^{\nu}} = C^k_{ij}u^{\mu}_k \qquad (2.3.22)$$

where

$$C^k_{ij} = \mu^r_i\mu^s_j\left(\frac{\partial\lambda^k_r}{\partial\theta^j} - \frac{\partial\lambda^k_s}{\partial\theta^r}\right) \qquad (2.3.23)$$

The m^3 quantities C^k_{ij} are independent from θ. This can be seen by differentiating Equation (2.3.22) with

respect to θ. After some simple calculations, one then see that

$$\frac{\partial C^k_{ij}}{\partial \theta^l} = 0 \qquad i,j,k,l=1,2,\ldots,m \qquad (2.3.24)$$

In this way we reach **Lie's second theorem**.

THEOREM 2.3.2. If X_i, $i=1,2,\ldots,m$ are the generators of an m-dimensional Lie group, they satisfy the closure relations

$$[X_i,X_j]_A = X_iX_j-X_jX_i = C^k_{ij}X_k \qquad (2.3.25)$$

where the quantities C^k_{ij} are called structure constants.

The symbol A in Equation (2.3.25) denotes an associative algebra with a conventional, associative product of operators X_iX_j. At closer inspection, this algebra emerges as being the *universal enveloping associative algebra* of the Lie algebra characterized by the rule (2.3.25).

The *fundamental Lie's rule* (2.3.25) can be explicitly written :

$$[X_i,X_j]_A = \left[u^\mu_i \frac{\partial}{\partial a^\mu} , u^\nu_j \frac{\partial}{\partial a^\nu} \right] = C^k_{ij}\, u^\sigma_k \frac{\partial}{\partial a^\sigma} \qquad (2.3.26)$$

where the product $[X_i,X_j]_A$ is Lie; that is, it satisfies the identities :

$$[X_i,X_j]_A + [X_j,X_i]_A = 0 \qquad (2.3.27a)$$

$$[[X_i,X_j]_A,X_k]_A + [[X_j,X_k]_A,X_i]_A + [[X_k,X_i]_A,X_j]_A = 0$$
(2.3.27b)

By substituting into these expressions the explicit form of the Lie product in terms of the structure constants, **Lie's third theorem** is reached.

THEOREM 2.3.3 The structure constants of a Lie group in standard realization (18) obey the relations :

$$C^k_{ij} + C^k_{ji} = 0 \qquad (2.3.28a)$$

$$C^k_{ij}C^r_{kl} + C^k_{jl}C^r_{ki} + C^k_{li}C^r_{kj} = 0 \qquad (2.3.28b)$$

Theorems (2.3.1-2-3) essentially provide the correspondence between a given (connected) lie goup G and its Lie algebra **G**. In particular, they allow the characterization of a Lie group in the neighborhood of the identity via the structure constants. We have here tacitly implied that different Lie groups may exist all admitting the same Lie algebra, that is, the same structure constants. However, among all Lie groups with the same Lie algebra only one is simply connected, called the **universal covering group**. For instance, group **SU(2)** (**SL(2.C)**) is the universal covering group of the group of rotations **SO(3)** (the homogeneous Lorentz group **SO(3.1)**).

The inverse transition from a Lie algebra to a corresponding Lie group can be characterized via the

inverses of Lie's first, second, and third theorems.

We pass now to the Lie-isotopic generalization of Definition 2.3.1 and Theorems (2.3.1-2-3) The prior identification of the main objective may be useful here. Lie's crucial result is fundamental rule (2.3.25). This rule essentially characterizes Lie algebras via the conventional associative product $X_i X_j$ of vector fields $X_i = u_i^\mu(a)\partial/\partial a^\mu$ on a manifold M. Our main objective is to generalized Definition 2.3.1 and Theorem (2.3.1-2-3) in such a way as to characterize a Lie algebra via the most general possible associative product $X_i * X_j$ of vector fields on a manifold.

Of utmost importance is the condition that the *local coordinates* a^μ, *the parameters* θ^i, *and the generators* X_i *of the conventional formulation of Lie's theorems are not changed in their isotopic generalization.* This is due to physical requirements which are uncompromisable for the description under consideration. As we recalled earlier, the local coordinates of M customerily have a direct physical meaning such as the coordinates of the frame of the experimental setup; the parameters carry a direct physical meaning as measurable quantities such as time, angle, etc., and the generators directly represent physical characteristics such as energy, angular momentum, etc. When the conventional description of self-adjoint interactions via Theorems (2.3. 1-2-3-3) is broadened to permit the additional presence of the non-self-adjoint interactions,the frame of the experi-

mental observer must be preserved; and physical characteristic such as energy and angular momentum must also be preserved unaltered.

These objectives can be realized as follows :

DEFINITION 3.2.2 Let

$$G : a^{\nu} \longrightarrow a^{*\mu}=g^{\mu}_{\nu}(a;\theta)f^{\nu}(a,\theta) = f^{*\mu}(a;\theta) \qquad (2.3.42a)$$

$$\det(g^{\mu}_{\nu})\neq 0 \ , \ g^{\mu}_{\nu}|_{\theta=0} = \delta^{\mu}_{\nu} \qquad (2.3.42b)$$

which verify the following properties :

(a) The transformations $a^{*}=f^{*}(a;\theta)$ constitute a Lie Transformation group, by therefore verifying conditions 1-5 of Definition 2.3.1.

(b) The group G^{*} is realized via the same base manifold, the same parameters and the same generators of G.

(c) When reduced in the neighborhood of the identity transformations, the group G^{*} can be characterized by the Lie algebra isotope $\mathbf{G}^{*}$ of $\mathbf{G}$.

Condition (c) is introduced to avoid non-Lie, Lie-admissible algebras in the neighborhood of the identity transformations. As a matter of fact, it is precisely this possibility that permits the further generalization of Lie's theory of type III (Santilli 1978a, 1982d).

Since the group of transformations $f^{*\mu}(a;\theta)$ is a

conventional, connected Lie group by assumption, it can be studied in the neighborhood of the identity as in the conventional case. The repetition of the analysis of $f(a;\theta)$ then yields the expressions :

$$da^{\mu} = u_k^{*\mu}(a)\lambda_i^k(\theta)d\theta^i \tag{2.3.43a}$$

$$u_k^{*\mu}(a) = \left| \frac{\partial}{\quad {}^{k}} g_{\nu}^{\mu}(a;\theta)f^{\nu}(a;\theta) \right| \tag{2.3.43b}$$

In order to realize the isotopy, we then introduce the following reformulation in terms of the quantities of G for given $g_k^i(a)$ functions :

$$u_{\nu}^{*\mu}(a) = g_k^i(a)u_i^{\mu}(a) \ , \quad \det(g_k^i) \neq 0 \tag{2.3.44}$$

Note that the other possibility $u_k^{*\mu} = g_{\nu}^{\mu}u_k^{\nu}$, even though conceivable (and actually more in line with Equations (2.3.43)), is excluded here because it would imply the redefinition of the generators :

$$X_k = u_k^{\mu}\left(\frac{\partial}{\partial a^{\mu}}\right) \longrightarrow X_k^* = g_{\nu}^{\mu}u_k^{\nu}\left(\frac{\partial}{\partial a^{\mu}}\right)$$

which is contrary to the notion of isotopy under study. The analyticity of the transformations then trivially implies the following generalization of Lie's First Theorem.

THEOREM 2.3.4.[6] (Santilli, loc. cit.) If transformations (2.3.42) characterize an isotopic image G^* of the Lie group G of transformations (2.3.41), then analytic functions $g^i_k(a)$ exist such that :

$$\frac{\partial a^{*\mu}}{\partial \theta^i} = g^k_i(a) u^\mu_k(a) \lambda^i_j \ , \quad \det g \neq 0 \qquad (2.3.45)$$

and the $u^\mu_k(a)$ functions are analytic.

This theorem, though analytic trivial, has nontrivial implications. Indeed, it implies a modification of the structure of the group in the neighborhood of the identity, i.e.,

$$G : a'^\mu \approx a^\mu + \theta^i u^\mu_i(a) \longrightarrow G^* : a^\mu + \theta^i g^j_i(a) u^\mu_j(a) \quad (2.3.46)$$

which is precisely the desired situation. We must now identity the integrability conditions under which such a

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(6) The *identity transformation* of a Lie group should not be confused with the *unit element* of the universal enveloping associative algebra. As we shall see, the identity trasnformation of G^* is preserved in a way compatible with the loss of the conventional unit element **1** for *A^*(G)*.

behavior is still Lie in algebraic character, when expressed in terms of the generators and parameters of the original group. Under these conditions, we say that the quantities g_i^j of Equations (2.3.45) or (2.3.46) are **isotopic functions** with respect to G.

The group G is Lie and thus admits the standard realization worked out earlier in this chart :

$$u_i^\nu \frac{\partial}{\partial a^\nu} u_j^\mu - u_j^\nu \frac{\partial}{\partial a^\nu} u_i^\mu = C_{ij}^k\, u_k^\mu \frac{\partial a}{\partial a^\mu} \tag{2.3.47a}$$

$$C_{ij}^k = \mu_i^r \mu_j^s \left(\frac{\partial \lambda_r^k}{\partial \theta^s} - \frac{\partial \lambda_s^k}{\partial \theta^r} \right) \tag{2.3.47b}$$

$$[X_i, X_j]_A = X_i X_j - X_j X_i = C_{ij}^k\, X_k \tag{2.3.47c}$$

$$X_k = u_k^\mu(a) \frac{\partial}{\partial a^\mu} \tag{2.3.47d}$$

The group G^* is also Lie and thus can be realized in the standard form :

$$u_i^{*\nu} \frac{\partial}{\partial a^\nu} u_j^{*\mu} - u_j^{*\nu} \frac{\partial}{\partial a^\nu} u_i^{*\mu} = C_{ij}^{*k}\, u_k^{*\mu} \frac{\partial a}{\partial a^\mu} \tag{2.3.48a}$$

$$C_{ij}^{*k} = \mu_i^{*r} \mu_j^{*s} \left(\frac{\partial \lambda_r^{*k}}{\partial \theta^s} - \frac{\partial \lambda_s^{*k}}{\partial \theta^r} \right) \tag{2.3.48b}$$

$$[X^*_i, X^*_j]_A = X^*_i X^*_j - X^*_j X^*_i = C^{*k}_{ij} X^*_k \tag{2.3.48c}$$

$$X^*_k = u^{*\mu}_k(a) \frac{\partial}{\partial a^\mu} \tag{2.3.48d}$$

However, this realization generally implies a change of the generators in the transition from G to G^* :

$$\mathbf{G} : X_k = u^\mu_k \frac{\partial}{\partial a^\mu} \longrightarrow G^* : X^*_k = u^{*\mu}_k \frac{\partial}{\partial a^\mu} \tag{2.3.49}$$

and, as such, does not verify the conditions for isotopy. To achieve the objective under consideration, we introduce the following isotopy of the universal enveloping associative algebra, realized via *functions on the base manifold* :

$$A(\mathbf{G}) : X_i X_j \longrightarrow A^*(\mathbf{G}) : X_i * X_j = g^r_i X_r g^s_j X_s \tag{2.3.50}$$

Notice that this mapping does verify the conditions of isotopy, in the sense that it is realized via the generators of the original algebra, while preserving the associativity of the product,

$$(g^r_i X_r g^s_j X_s) g^t X_t = g^r_i X_r (g^s_j X_s g^t_k X_t) \tag{2.3.51}$$

The fundamental Lie rule (2.3.47c) can now be written :

$$u^\nu_i \frac{\partial}{\partial a^\nu} * u^\mu_j - u^\nu_j \frac{\partial}{\partial a^\nu} * u^\mu_i = \mathfrak{C}^k_{ij} u^\mu_k \tag{2.3.52a}$$

$$\bar{C}^k_{ij} = C^{*r}_{ij} g^k_r(a) \qquad (2.3.52b)$$

The integrability conditions for the functions $g^i_k(a)$ to be isotopic, that is, to yield rule (2.3.52), can then be readily computed. Thus we reach the following generalization of Lie's second theorem.

THEOREM 2.3.5. (Santilli, loc. cit.) Under the integrability conditions :

$$g^k_i u^\nu_k \frac{\partial}{\partial a^\nu} g^l_j - g^k_j u^\nu_k \frac{\partial}{\partial a^\nu} g^l_i = g^r_j g^s_i C^l_{rs} + C^{*r}_{ij} g^l_k \qquad (2.3.53)$$

the generators X_i of an isotope G^* of a Lie group G satisfy the isotopic rule of associative Lie-admissibility :

$$[X_i, X_j]_A = X_i * X_j - X_j * X_i = \bar{C}^k_{ij}(a) X_k \qquad (2.3.54a)$$

$$A^*(G) : X_i * X_j = g^r_i X_r g^s_j X_s \qquad (2.3.54b)$$

$$X_k = u^\mu_k(a) \frac{\partial}{\partial a^\mu} \qquad (2.3.54c)$$

where the quantities $\bar{C}^k_{ij}(a)$, here called **structure functions**, are generally dependent on the (local) coordinates of the base manifold of the original group.

In this way we reach an interpretation of the F^*-linear combination of the isotopically mapped standard monomials. While in the standard realization (2.3.47c) the quantities C^{*k}_{ij} are constants (the structure constants of a Lie group), the corresponding quantities which emerge after the reformulation of the same group G^* in terms of the base manifold, the parameters, and the generators of G, acquire an explicit dependence on the local coordinates (the structure functions $C^{*k}_{ij}(a)$). This situation has numerous technical implications (e.g., from the viewpoint of the representation and classification theory) which are not considered here.

The reformulation of Lie's third theorem is now straightforward. Indeed, the use of the Lie algebra laws for the isotopically mapped product (2.3.54a) yields the following property.

THEOREM 6. (Santilli, loc. cit.). The structure functions $C^{*k}_{ij}(a)$ of the isotopic realization of a Lie group G^* verify the identities :

$$\tilde{C}^k_{ij} + \tilde{C}^k_{ij} = 0$$

$$\tilde{C}^k_{ij}\,\tilde{C}^r_{kl} + \tilde{C}^k_{jl}\,\tilde{C}^r_{ki} + \tilde{C}^k_{li}\,\tilde{C}^r_{kj} + [\tilde{C}^r_{ij}, X_l]_{A^*} + [\tilde{C}^r_{jl}, X_i]_{A^*} + + [\tilde{C}^r_{li}, X_j]_{A^*} = 0 \qquad (2.3.55b)$$

The exponentiation from the Lie algebra to the Lie group can

now be formulated in terms of the **isotoric image of the exponential law** (2.3.37), i.e.,

$$G : \exp(\theta i X_i)\Big|_A \longrightarrow G^* : \exp(\theta i X_i)\Big|_{A^*}, \quad (2.3.56)$$

which is based on the following rule of Lie isotopy :

$$G : [X_i, X_j]_A = C_{ij}^k X_k \longrightarrow G^* : [X_i, X_j]_{A^*} = \bar{C}_{ij}^k(a) X_k \quad (2.3.57)$$

with consequential isotopically mapped Baker-Campbell-Hausdorff formula :

$$\exp(X_\alpha^*)\exp(X_\beta^*) = \exp(X_j^*) \quad X^* = gX \quad (2.3.58a)$$

$$X^* = X_\alpha^* + X_\beta^* + \frac{1}{2}[X_\alpha, X_\beta]_{A^*} + \frac{1}{12}\big[(X_\alpha - X_\beta), [X_\alpha, X_\beta]_{A^*} + \ldots \quad (2.3.58b)$$

whose existence is ensured by that of the standard realization. The reader can now see the emergence of the F^*-linear combination of the basis directly in the group composition law. Clearly, the enveloping algebra underlying expressions (2.3.57) is the isotope $A^*(G)$ of $A(G)$.

CHAPTER III

ISOMANIFOLDS

The extension of the isotopies to continuous structure was initialed with the notion of isodifferential calculus (Santilli 1991a). In this and in the following chapter we shall review the studies by the authors (Sourlas and Tsagas 1992a,b) on the isotopies of manifolds and their primary properties.

3.1. REAL CARTECIAN MANIFOLD.

DEFINITIONS 3.1.1 : Let

$$\mathbb{R}^n = \left\{ (x_1,\ldots,x_n) \;/\; x_i \in \mathbb{R},\ i=1,\ldots,n \right\}$$

be the conventional n-dimensional Cartecian space. Then on $\mathbb{R}^n$ we can define the following structures :

3.1.1 <u>Vector structure</u> : On $\mathbb{R}^n$ we define two operations as follows :

(i) $+ : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$$+ : \{(x_1,\ldots,x_n),(y_1,\ldots,y_n)\} \longmapsto (x_1,\ldots,x_n) + (y_1,\ldots,y_n) \equiv (x_1+y_1,\ldots,x_n+y_n)$$

(ii) $\cdot : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$\cdot : \{\lambda,(x_1,\ldots,x_n)\} \longmapsto \lambda(x_1,\ldots,x_n) \equiv (\lambda x_1,\ldots,\lambda x_n)$

Then $\mathbb{R}^n$ with the two operations $(+,\cdot)$ becomes a vector space over $\mathbb{R}$ of dimension n, which is called **real Cartesian vector space of dimension n**, denoted by $V^n(\mathbb{R})=\{\mathbb{R}^n,+,\cdot\}$, whose base is :

$\{\ e_1=(1,0,0,\ldots,0),\ e_2=(0,1,0,\ldots,0),\ldots,\ e_n=(0,0,0,\ldots,1)\}$

called **canonical base**.

3.1.2 Affine structure : It is well known that to the vector space $V^n(\mathbb{R})=\{\mathbb{R}^n,+,\cdot\}$ we can associate a set $A\equiv\mathbb{R}^n$ if there exist a mapping f :

$$f : A\times A \longrightarrow V^n(\mathbb{R})$$
$$f : (P,Q) \longmapsto F(P,Q)=v\in V^n(\mathbb{R})$$

with the following properties :

i) $(\forall P_1,P_2,P_3\in A)\left[\ P_1P_3 = P_1P_2+P_2P_3\ \right]$

ii) $(\forall P\in A)\,(\forall v\in V^n(\mathbb{R}))\,(\exists Q\in A)\left[\ PQ = v\ \right]$

The set A is called **real Cartesian affine space of dimension n** and will be denoted $A^n(\mathbb{R})$.

We consider the points :

$P_0=(0,0,0,\ldots,0),\ \ P_1=(1,0,0,\ldots,0),\ \ P_2=(0,1,0,\ldots,0),\ldots$
$P_n=(0,0,0,\ldots,1)$

of the real Cartesian affine space $A^n(\mathbb{R})$. These points form a base of A^n, because the vectors :

$$e_1=P_0P_1=(1,0,\ldots,0), \qquad e_2=P_0P_2=(0,1,\ldots,0),\ldots,$$
$$e_n=P_0P_n=(0,0,\ldots 1)$$

form the canonical base of $V^n(\mathbb{R})$. This base $\{P_0,P_1,\ldots,P_n\}$ is called **fundamental base** of $A^n(\mathbb{R})$.

3.1.3 Affine coordinates of a point

If $P\in A^n(\mathbb{R})$, then $PP_0\in V^n(\mathbb{R})$, which can be written :

$$PP_0=o_1e_1+\ldots+o_ne_n$$

The numbers $(o_1,\ldots,o_n)$ are called **affine coordinates** of the point P with respect to the fundamental base $\{P_0,P_1,\ldots,P_n\}$ of $A^n(\mathbb{R})$.

3.1.4 Natural affine coordinate functions on $A^n(\mathbb{R})$

On the real cartesian affine space $A^n(\mathbb{R})$ consider the function x_i, $i=1,\ldots,n$, defined by :

$$x_i : A^n(\mathbb{R}) \longrightarrow \mathbb{R}$$
$$x_i : P=(o_1,\ldots,o_n) \longmapsto x_i(P)=o_i$$

These functions are called **natural affine coordinate**.

3.1.5 Topological structure.

On the set $\mathbb{R}^n$ we consider a topology T defined by :

$$T=\left\{ \emptyset \ , \ \mathbb{R}^n, \ \bigcup_{i\in I} B_i \right\}$$

where B_i, the subset of $\mathbb{R}^n$, is defined by :

$$B_i = \left\{ P=(p_1,\ldots,p_n) \ \ / \ \ o_1<p_1<\beta_1,\ldots,a_n<p_n<\beta_n, \ o_i,\beta_i\in\mathbb{R} \right\}$$

This topology T can be considered as the Cartesian product of the topology of the open intervals on the straight line n times. The topological space $\{\mathbb{R}^n,T\}$ is denoted by $T^n(\mathbb{R})$ and called **real cartesian topological space.**

REMARK 3.1.1. The real cartesian topological space $T^n(\mathbb{R})$ is a Hausdorf space

3.1.6 Cartesian manifold, Let

$$\mathbb{R}^n=\{P=(x_1,\ldots,x_n)/x_i\in\mathbb{R} \ i=1,\ldots,n\}$$

be the cartesian product of $\mathbb{R}$ n times. On this space we consider the previous structures, that is the vector structure, the affine structure and the topological structure. We also consider the mapping :

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^n \ , \qquad f : P \longmapsto f(P)=P \quad \forall P\in\mathbb{R}^n$$

which is the identity on $\mathbb{R}^n$. This set $\mathbb{R}^n$ with the three

structures and the mapping f is denoted by :

$$M^n(\mathbb{R})=\{\mathbb{R}^n, f\}$$

and is called **real Cartesian manifold of dimension n.**

3.2. REAL CARTESIAN ISOMANIFOLD

DEFINITION 3.2.1 We consider the real isofield $\hat{\mathbb{R}}$, which is an isotope of $\mathbb{R}$ with the new multiplicative unit $\hat{I}$, called **multiplicative isounit**. It is known that $\hat{\mathbb{R}}$ is defined by :

$$\hat{\mathbb{R}} = \{ \hat{o} \;/\; \hat{o}=o\hat{I} \;,\; o\in\mathbb{R} \text{ and } \hat{I}\neq 0\}$$

We consider the Cartesian product of $\hat{\mathbb{R}}$ with itself n times, then we obtain the space :

$$\hat{\mathbb{R}}^n = \{(\hat{o}_1,\ldots,\hat{o}_n) \;/\; \hat{o}_i\in\hat{\mathbb{R}} \;,\; i=1,\ldots,n \}$$

which is called **real Cartesian isospace** (Sourlas and Tsagas, 1992a first paper at AGC). On $\hat{\mathbb{R}}^n$ we can define the following structures :

3.2.1 __Vector structure.__ From the vector space $V^n(\mathbb{R})$ if we use the isofield $\hat{\mathbb{R}}$, we obtain the isovector space, which is denoted by $V^n(\hat{\mathbb{R}})$ and called **real Cartesian isovector space.** The vectors :

$$e_1=(1,0,\ldots,0) \;,\; e_2=(0,1,\ldots,0), \;\ldots\;, e_n=(0,0,\ldots 1)$$

which form the canonical base of $V^n(\mathbb{R})$, are also a base of $V^n(\hat{\mathbb{R}})$. Every isovector $v\in V^n(\hat{\mathbb{R}})$ can be written :

$$v=\hat{\lambda}_1 e_1+\hat{\lambda}_2 e_2+\ldots+\hat{\lambda}_n e_n \quad , \quad \hat{\lambda}_i \in \hat{\mathbb{R}} \quad i=1,\ldots,n$$

3.2.2 <u>Change of the base in $V^n(\hat{\mathbb{R}})$</u>. Let $\{t_1,\ldots,t_n\}$ be a base of $V^n(\hat{\mathbb{R}})$ different than the canonical base $\{e_1,\ldots,e_n\}$. We decompose the vectors $t_1,\ldots,t_n$ with respect to the base $\{e_1,\ldots,e_n\}$ and then we have :

$$\left.\begin{array}{l} t_1 = \hat{\lambda}_1^1 e_1+\hat{\lambda}_1^2 e_2+\ldots+\hat{\lambda}_1^n e_n \\ t_2 = \hat{\lambda}_2^1 e_1+\hat{\lambda}_2^2 e_2+\ldots+\hat{\lambda}_2^n e_n \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ t_1 = \hat{\lambda}_n^1 e_1+\hat{\lambda}_n^2 e_2+\ldots+\hat{\lambda}_n^n e_n \end{array}\right\} \qquad (3.2.1)$$

where $\hat{\lambda}_j^i \in \hat{\mathbb{R}}$, i=1,...,n. Eqs (3.2.1) can be written

$$\begin{pmatrix} t_1 \\ t_2 \\ \cdot \\ \cdot \\ t_n \end{pmatrix} = \begin{pmatrix} \hat{\lambda}_1^1 & \hat{\lambda}_1^2 & \ldots\ldots & \hat{\lambda}_1^n \\ \hat{\lambda}_2^1 & \hat{\lambda}_2^2 & \ldots\ldots & \hat{\lambda}_2^n \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \hat{\lambda}_n^1 & \hat{\lambda}_n^2 & \ldots\ldots & \hat{\lambda}_n^n \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \cdot \\ \cdot \\ e_n \end{pmatrix}$$

Therefore, the matrix :

$$(\hat{\lambda}_i^j) = \begin{pmatrix} \hat{\lambda}_1^1 & \hat{\lambda}_1^2 & \cdots\cdots & \hat{\lambda}_1^n \\ \hat{\lambda}_2^1 & \hat{\lambda}_2^2 & \cdots\cdots & \hat{\lambda}_2^n \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \hat{\lambda}_n^1 & \hat{\lambda}_n^2 & \cdots\cdots & \hat{\lambda}_n^n \end{pmatrix} \in GL(n,\hat{\mathbb{R}})$$

is the transpose matrix from one base to another, where $GL(n,\hat{\mathbb{R}})$ is the set of all invertible square matrices of order n with elements in the isofield $\hat{\mathbb{R}}$ of $\mathbb{R}$, is the transpose matrix from one base to another.

3.2.3 Affine structure. We can associate to the vector space $V^n(\hat{\mathbb{R}})$ the affine space $A^n(\hat{\mathbb{R}})$ which, as a set is identified with $\hat{\mathbb{R}}^n$. Then $A^n(\hat{\mathbb{R}})$ is called **real Cartesian isoaffine space of dimension n.** $A^n(\hat{\mathbb{R}})$ is isotopic to $A^n(\mathbb{R})$.

Let $P_0(0,0,0,\ldots,0)$, $P_1(0,1,0,\ldots,0)$, $P_2(0,0,1,\ldots,0)$, $\ldots P_n(0,0,0,\ldots 1)$ be n+1 points of $A^n(\hat{\mathbb{R}})$. These points form an affine base of $A^n(\hat{\mathbb{R}})$ because the vectors :

$$P_0P_1 \ , \ P_0P_2 \ , \ldots, \ P_0P_n$$

form a base of $A^n(\hat{\mathbb{R}})$, which is called **fundamental affine base** of $V^n(\hat{\mathbb{R}})$.

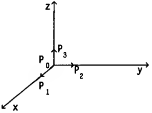
3.2.4 Isoaffine coordinates of a point. If P is a point of $A^n(\hat{\mathbb{R}})$, then $P_0P \in V^n(\hat{\mathbb{R}})$, which can be written :

$$PP_0=\beta_1 P_0P_1+\beta_2 P_0P_2+\ldots+\beta_n P_0P_n$$

where $\beta_1,\beta_2,\ldots,\beta_n\in\hat{\mathbb{R}}$, which are called **isoaffine coordinates** of the point P with respect to fundamental affine base of $A^n(\hat{\mathbb{R}})$.

EXAMPLE 3.2.1 Let $A^3(\hat{\mathbb{R}})$ be the real cartesian isoaffine space of three dimension. Determine grafically its fundamental affine base.

Solution : The points $P_0(0,0,0)$, $P_1(1,0,0)$, $P_2(0,1,0)$ and $P_3(0,0,1)$, which are essentially the origin and the



coordinates of the unit vectors on the axes OX, OY and OZ respectively, form the fundamental affine base, because the vectors :

$$P_0P_1\ ,\ P_0P_2,\ P_0P_3$$

form the canonical base of $V^3(\hat{\mathbb{R}})$

3.2.5 Natural isoaffine coordinate function. On the real Cartesian space $A^n(\hat{\mathbb{R}})$ we consider the functions $\hat{x}_i$, i=1,...n

defined by :

$$\hat{x}_i : A^n(\hat{\mathbb{R}}) \longrightarrow \hat{\mathbb{R}}$$

$$\hat{x}_i : P=(p_1,\dots,p_n) \longmapsto \hat{x}_i(P) \equiv p_i \ , \ i=1,\dots,n$$

are called **natural isoaffine coordinate.**

3.2.6 Change of bases in $A^n(\hat{\mathbb{R}})$. On the $A^n(\hat{\mathbb{R}})$ we can define other affine bases except the fundamental. We consider another base :

$$\{Q_0,Q_1,\dots,Q_n\}$$

of $A^n(\hat{\mathbb{R}})$, which always contains n+1 points. In the same way we can define isoaffine coordinate functions $y_1,y_2,\dots,y_n$ on $A^n(\hat{\mathbb{R}})$ with respect to the base $\{Q_0,Q_1,\dots,Q_n\}$ as we have defined the natural isoaffine coordinate functions with respect to the fundamental affine base. The functions $(y_1,y_2,\dots,y_n)$ are related with $(x_1,x_2,\dots,x_n)$ by the relations :

$$\begin{aligned} y_1 &= a_{11}x_1+a_{12}x_2+\dots+a_{1n}x_n+a_1 \\ y_2 &= a_{21}x_1+a_{22}x_2+\dots+a_{2n}x_n+a_2 \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ y_n &= a_{n1}x_1+a_{n2}x_2+\dots+a_{nn}x_n+a_n \end{aligned}$$

which can be written as :

$$\begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ a_n \end{pmatrix}$$

where

$$a = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \in GL(n,\hat{\mathbb{R}})$$

3.2.7 Topological structure. On the set $\hat{\mathbb{R}}^n$, which coincides with the set $\mathbb{R}^n$, we consider the same topology on $\hat{\mathbb{R}}^n \equiv \mathbb{R}^n$ as sets, which has been defined in 3.1.5. The set $\hat{\mathbb{R}}^n = \mathbb{R}$ with the topology T is called **real Cartesian isotopological space** and denoted by $T^n(\hat{\mathbb{R}})$. It is obvious that :

$$T^n(\mathbb{R}) \equiv T^n(\hat{\mathbb{R}})$$

3.2.8 Cartesian isomanifold. Let $\hat{\mathbb{R}}^n = \{(x_1,\ldots,x_n)/x_i \in \hat{\mathbb{R}}, i=1,\ldots,n\}$ be the real isocartesian space of dimension n. On this space we obtain the previous structures, that is, the

vector structure, the affine structure and the topological structure. We consider the mapping :

$$f : \hat{\mathbb{R}}^n \longrightarrow \hat{\mathbb{R}}^n \quad , \quad f : P \longmapsto f(P)=P \ \forall P\in\hat{\mathbb{R}}$$

which is the identity on $\hat{\mathbb{R}}$. The set $\hat{\mathbb{R}}^n$, with the three structures and the mapping f, is called **real Cartesian isomanifold** of dimension n.

3.3 ISOMANIFOLD

DEFINITION 3.3.1 (Sourlas and Tsagas 1992a) : Let M be a Hausdorff space. We consider the pair $(U_\alpha,\varphi_\alpha)$, where $U_\alpha\subseteq M$ and φ_α is a homeomorphism of U_α onto an open subset V_α of $\hat{\mathbb{R}}^n$, that is :

$$\varphi_\alpha : U_\alpha \longrightarrow \varphi_\alpha(U_\alpha)=V_\alpha\subseteq\hat{\mathbb{R}}^n$$

The pair $(U_\alpha,\varphi_\alpha)$ is called **isochart** on M. From the above we conclude that the isochart $(U_\alpha,\varphi_\alpha)$ has the following meaning. The subset $U_\alpha\subseteq M$, since it is homeomorphic onto a subset of $M^n(\hat{\mathbb{R}})$, which is the real cartesian isomanifold of dimension n, is reduced to the study of $M^n(\hat{\mathbb{R}})$.

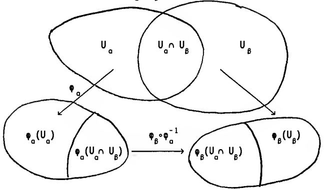
Let $(U_\alpha,\varphi_\alpha)_{\alpha\in A}$ be the set of all isocharts on M with the following three properties :

P_1 $\quad \bigcup\limits_{\alpha\in A} U_\alpha = M$

P_2 $\quad$ We consider the two isocharts $(U_\alpha,\varphi_\alpha)$ and (U_β,φ_β)

with the condition :

$$U_\alpha \cap U_\beta \neq 0$$



The mapping $\varphi_\beta \circ \varphi_\alpha^{-1}$ is the following :

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta) \qquad (3.3.1)$$

Relation (3.3.1) is obtained as follows :

$$\varphi_\alpha : Z=(U_\alpha \cap U_\beta) \longrightarrow \varphi_\alpha(Z)=\varphi_\alpha(U_\alpha \cap U_\beta)$$

$$\varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow Z=U_\alpha \cap U_\beta \qquad (3.3.2)$$

$$\varphi_\beta : Z=(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(Z)=\varphi_\beta(U_\alpha \cap U_\beta)$$

which is Eq. (3.3.1).

P_3 The collection $(U_\alpha,\varphi_\alpha)_{\alpha\in A}$ contains the maximal number of isocharts satisfying the above conditions.

We assume that all the mappings (3.3.1) are class C^k, that is, they are continously k times differentiable.

One collection of isocharts $(U_\alpha,\varphi_\alpha)_{\alpha\in A}$, which satisfies the above three properties P_1, P_2 and P_3 is called **a differentiable isoatlas** of class C^k on M.

REMARK 3.3.2 : The mapping $\varphi_\beta\circ\varphi_\alpha^{-1}$ maps the $\varphi_\alpha(U_\alpha\cap U_\beta)$ into $\varphi_\beta(U_\alpha\cap U_\beta)$, that is :

$$\varphi_\beta\circ\varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha\cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha\cap U_\beta)$$

Both sets $\varphi_\beta(U_\alpha\cap U_\beta)$ and $\varphi_\beta(U_\alpha\cap U_\beta)$ are subsets of $\mathbb{R}^n$. Hence $\varphi_\beta\circ\varphi_\alpha^{-1}$ can be expressed by :

$$\varphi_\beta\circ\varphi_\alpha^{-1} = \begin{cases} y_1=f_1(x_1,\dots,x_n) \\ \\ y_n=f_n(x_1,\dots,x_n) \end{cases}$$

where $(x_1,x_2,\dots x_n)$ and $(y_1,y_2,\dots,y_n)$ two isoaffine coordinate functions on $\hat{\mathbb{R}}^n$ with respect to two affine bases $\{S_0,S_1,\dots,S_n\}$ and $\{Q_0,Q_1,\dots,Q_n\}$ respectively of $\hat{\mathbb{R}}^n$.

DEFINITION 3.3.1 If the mapping $\varphi_\beta\circ\varphi_\alpha^{-1}$, for all the pairs (α,β) for which $U_\alpha\cap U_\beta\neq\emptyset$,are differentiable of class C^k, or briefly C^k, then the isoatlas $(U_\alpha,\varphi_\alpha)_{\alpha\in A}$ on M is called

isodifferential structure of class C^k on M and denoted by (U_o,φ_o) $o\in A$ C^k.

DEFINITION 3.3.2 Let M be a Hausdorff space on which we consider an isoatlas (U_o,φ_o) $o\in A$, C^k. The pair $\{M,(U_o,\varphi_o)o\in A\}$ is called **differential isomanifold of dimension n and class k.**

REMARK 3.3.3 From now on when we say differentiable isomanifold we mean of class C^∞, in any other case we shall state.

DEFINITION 3.3.3 Let $\{M,(U_o,\varphi_o)o\in A\}$ be a differential isomanifold of dimension n. It is known that :

$$\varphi_o : U_o \longrightarrow \varphi_o(U_o) \subseteq \hat{\mathbb{R}}^n$$

$$\varphi_o : q \longmapsto \varphi_o(q)=(x_1(q),\ldots,x_n(q))\in\hat{\mathbb{R}}^n$$

The coordinates $(x_1(q),\ldots,x_n(q))$, $q\in U_o$ are called **iso-coordinates of the isochart** (U_o,φ_o).

EXAMPLE 3.3.1 Determine an isoatlas on the cartesian isomanifold $M^n(\hat{\mathbb{R}})$.

<u>Solution</u> : It is known that $M^n(\hat{\mathbb{R}})\equiv\mathbb{R}^n$, with the mentioned structures, is a Haudorff space. We consider an isoatlas on $M^n(\hat{\mathbb{R}})$ contain only one isochart $(U_o\equiv\hat{\mathbb{R}}^n,\varphi_o)$ where

$$\varphi_o : U_o\equiv\hat{\mathbb{R}} \longrightarrow \hat{\mathbb{R}}^n$$

$$\varphi_a : P \longmapsto \varphi_a(P)=P \quad \forall P \in \mathbb{R}^n$$

So $M=\mathbb{R}^n$, with the isoatlas (U_a,φ_a), where $U_n \equiv \hat{\mathbb{R}}^n$ and φ_a is the identity map on $\hat{\mathbb{R}}^n$, is an isomanifold, which was called Cartesian isomanifold.

EXAMPLE 3.3.2 Let $M=\hat{\mathbb{R}}$ with $\left(U_a=\hat{\mathbb{R}},\ \varphi_a: x \longrightarrow x^3\right)$ be an isomanifold. Determine its differentiability.

Solution : The topological space $M=\hat{\mathbb{R}}$ is a Hausdorff space. Its isoatlas has only one isochart :

$$(U_a=\hat{\mathbb{R}},\ \varphi_a : x \longrightarrow x^3)$$

since we require :

$$\varphi_a^{-1} : x \longmapsto \varphi_a^{-1}(x)=y= \sqrt[3]{x}$$

The function $y=\sqrt[3]{x}$ is continuous and differentiable at $x=0$, but its derivative $y' = \frac{1}{3}\,\frac{1}{x^{2/3}}$ is not continous at $x=0$. Hence the isomanifold

$$\{\ \hat{\mathbb{R}}^n,\ (U_a=\hat{\mathbb{R}}^n\ ,\ \varphi_a : x \longrightarrow \sqrt[3]{x}\ \}$$

is of class C^0

PROBLEM 3.3.1 Let M be an isomanifold of dimension n. Does M admit always a differentiable structure ?

Solution : There are Hausdorff spaces which do not admit any differentiable structure C^k. With the same method we prove that there are Hausdorff spaces, which do not admit

any isostructures of any class C^r r=0,1,...

3.3.2 Different isostructures. Let M be a Hausdorff space. We assume that M admits an isodefferentiable structure. Is this a unique ?

It has been proved that there are manifolds which admit more than one differentiable structures. We refer to the sphere S^n $n\geq 7$, which can carry more than one differentiable structures. However this number of differentiable structures is finite. From the real Cartesian manifolds $\mathbb{R}^n$ $n\geq 1$ only $\mathbb{R}^4$ can carry more than one differentiable structures.

The same problem can be asked for the isodifferentiable structure. Hence on S^n $n\geq 7$ there are more than one isodifferentiable structures. On the real Cartesian isomanifold $\hat{\mathbb{R}}^n$ $n\geq 1$ there is only one isodifferentiable structure except $\hat{\mathbb{R}}^4$ on which there are more than one isodifferentiable structures

3.3.3 Analytic isomanifold. Let M be a Hausdorff space on which we consider an isoatlas $(U_\alpha, \varphi_\alpha)$ $\alpha\in A$ with the following three properties :

P_1 $\bigcup_{\alpha\in A} U_\alpha = M$

P_2 The mappings :

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta)$$

are analytic for all isocharts $(U_\alpha,\varphi_\alpha)$ $\alpha\in A$ with the properties P_1 and P_2 is the maximal possible.

The Hausdorff space M with the isoatlas $(U_\alpha,\varphi_\alpha)$ $\alpha\in A$ with the three properties is called **analytic isomanifold**.

REMARK 3.3.3. Every analytic isomanifold is a differentiable isomanifold. The inverse is not true. Therefore the set of analytic isomanifolds is a proper subset of the set of differentiable isomanifolds.

DEFINITION 3.3.4. Let M, $(U_\alpha,\varphi_\alpha)$ $\alpha\in A$ be an isomanifold. We consider two isocharts $(U_\alpha,\varphi_\alpha)$ and (U_β,φ_β) of the isoatlas $(U_\alpha,\varphi_\alpha)$ of the isoatlas $(U_\alpha,\varphi_\alpha)$ $\alpha\in A$ such that $U_\alpha\cap U_\beta\neq\varnothing$. The mapping $\varphi_\beta\circ\varphi_\alpha^{-1}$ is defined by :

$$\varphi_\beta\circ\varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha\cap U_\beta) \subseteq \hat{R}^n \longrightarrow \varphi_\beta(U_\alpha\cap U_\beta) \subseteq \hat{R}^n$$

where n is the dimension of the isomanifold .

On the open subsets $\varphi_\alpha(U_\alpha\cap U_\beta)$ and $\varphi_\beta(U_\alpha\cap U_\beta)$ we consider the coordinate systems :

$$(x_1,\dots,x_n) \text{ and } (y_1,\dots,y_n)$$

respectively. Therefore $\varphi_\beta\circ\varphi_\alpha^{-1}$ can be expressed by the

functions :

$$\varphi_\beta \circ \varphi_\alpha^{-1} = \begin{cases} y_1 = f_1(x_1,\ldots,x_n) \\ \ldots\ldots\ldots\ldots\ldots \\ y_n = f_n(x_1,\ldots,x_n) \end{cases} \tag{3.3.4}$$

which are the coordinate functions of the mapping $\Phi_\beta \circ \varphi_\alpha^{-1}$. From (3.3.4) we obtain the Jacobian determinant :

$$J_{\alpha\beta} = \frac{D(f_1,\ldots,f_n)}{D(x_1,\ldots,x_n)} = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_n} \\ \ldots\ldots & \ldots\ldots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_n} \end{vmatrix}$$

If $J_{\alpha\beta}>0$ $\forall(x_1,\ldots,x_n)\in\varphi_\alpha(U_\alpha\cap U_\beta)$ and for all pairs of isocharts $(U_\alpha,\varphi_\alpha)$, (U_β,φ_β) for which $U_\alpha\cap U_\beta\neq\emptyset$, then the isomanifold M is called **orientable.**

PROBLEM 3.3.2. Let M, $(U_\alpha,\varphi_\alpha)$ and N, (V_β,ψ_β) be two isomanifolds of dimension m and n respectively. The topological product MxN can be an isomanifold.

Proof : The topological product MxN is also a Hausdorff space. We consider the isoatlas :

$$(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta) \ (\alpha,\beta)\in A\times B$$

on MxN because it has the three properties :

$$P_1 \quad \bigcup_{(\alpha,\beta)\in A\times B} U_\alpha \times V_\beta = M\times N$$

From which the following property holds :

$$\varphi_\alpha \times \psi_\beta : U_A \times V_\beta \longrightarrow (\varphi_\alpha \times \psi_\beta)(U_\alpha \times V_\beta) = \varphi_\alpha(U_\alpha)\times\psi_\beta(V_\beta) \subseteq \mathbb{R}^{n+m}$$

P_2 For the two isocharts

$$(U_{\alpha_1} \times V_{\beta_1}, \varphi_{\alpha_1} \times \psi_{\beta_1}) , (U_{\alpha_2} \times V_{\beta_2}, \varphi_{\alpha_2} \times \psi_{\beta_2}) ,$$

of this isoatlas, for which

$$(U_{\alpha_1} \times V_{\beta_1}) \cap (U_{\alpha_2} \times V_{\beta_2}) \neq \varnothing$$

we have the mapping :

$$(\varphi_{\alpha_2} \times \psi_{\beta_2}) \circ (\varphi_{\alpha_1} \times \psi_{\beta_1})^{-1} : \varphi_{\alpha_1} \times \psi_{\beta_1} ((U_{\alpha_1} \times V_{\beta_1}) \cap (U_{\alpha_2} \times V_{\beta_2}))$$

$$\longrightarrow \varphi_{\alpha_2} \times \psi_{\beta_2} ((U_{\alpha_1} \times V_{\beta_1}) \cap (U_{\alpha_2} \times V_{\beta_2}))$$

P_3 The collection of these isocharts of the isoatlas

$$(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta) \quad (\alpha,\beta)\in A\times B$$

is maximal , satisfying properties P_1 and P_2.

Therefore, the Hausdorff space MxN with the mentioned isoatlas becomes one differentiable isomanifold of dimension n+m and it is called the **topological product** of the two differentiable isomanifolds M and N.

EXAMPLE 3.3.3. Let S^1 be an isosphere of one dimension,

which coincides with circumference. We consider the topological product of S^1 by itself k times. Then we obtain the isomanifold :

$$T^k = \underbrace{S^1 x S^1 x \ldots x S^1}_{k \text{ times}}$$

which is called **isotorus** of dimension k.

REMARK 3.3.4. Let $M^n(\hat{\mathbb{R}}) \equiv \hat{\mathbb{R}}^n$ be the Cartesian isomanifold of dimension n. We know that $\hat{\mathbb{R}}^n = M^n(\hat{\mathbb{R}})$ has avector structure, an affine structure and a topological structure. We consider the canonical base :

$$e_1 = (1,0,\ldots,0),\ e_2 = (0,1,\ldots,0)\ ,\ldots,\ e_n = (0,0,\ldots,1)$$

of $\hat{\mathbb{R}}$. and obtain the subset $\hat{\mathbf{Z}}^n$ of $\hat{\mathbb{R}}^n$, defined by :

$$\hat{\mathbf{Z}}^n = \{ \lambda_1 e_1 + \ldots + \lambda_n e_n \ /\ \lambda_1, \ldots, \lambda_n \in \hat{\mathbf{Z}} \}$$

The quotient space : $\hat{\mathbb{R}}^n / \hat{\mathbf{Z}}^n$ can be coincide with :

$$T^n = \underbrace{S^1 x \ldots x S^1}_{n \text{ times}}$$

Cartesian isocylinder 3.3.17. We consider the Cartesian isomanifold $\hat{\mathbb{R}}^n$ and obtain k<n vectors :

$$t_1, t_2, \ldots, t_k$$

We construct the subset $\hat{\mathbf{Z}}^k$ of $\hat{\mathbb{R}}^n$ as follows :

$$\hat{\mathbf{Z}}^k = \{ \lambda_1 t_1 + \ldots + \lambda_k t_k \ /\ \lambda_1, \ldots, \lambda_k \in \hat{\mathbf{Z}} \}$$

The quotient space : $C^k = \mathbb{R}^n / \hat{\mathbf{Z}}^k$ is called **Cartesian isocylider.**

CHAPTER IV

ISOTENSOR FIELDS ON ISOMANIFOLDS

In the preceding chapter we have introduced the notion of isomanifold and related topology that is applicable under isotopy.

In this chapter we shall study isovector and isotensor fields on isomanifolds (Sourlas and Tsagas 1992b). The formers are important for the characterization of the foundamental isotopy of Lie algebra (Santilli, 1978a). The latter are important for the condtruction of isotopic geometries (Santilli, 1991b).

4.I ISOFUNCTIONS

DEFINITION 4.I.1 Let M be a differentiable isomanifold where we have substituted the field of real number with its isofield $\hat{\mathbb{R}}$.

We consider the mapping :

$$f : M \longrightarrow \hat{\mathbb{R}} \ , \ f : P \longmapsto f(P)$$

which is called **isofunction** on the differentiable isomanifold.

DEFINITION 4.I.2 Let f be an isofunction on the differentiable isomanifold. The function f is called **diffe-**

rentiable of class C^k at the point P if, for every neighborhood V of P homeomorphic onto an open subset B of $\hat{\mathbb{R}}^n$ with corresponding homeomorphism :

$$\varphi : U \longrightarrow B \subseteq \hat{\mathbb{R}}^n$$

the function $f \circ \varphi^{-1}$ on B is k times differentiable. The above can be explained as follows :

Since U is a neighborhood of M homeomorphic onto an open subset of $\hat{\mathbb{R}}^n$, there exists one homeomorphism :

$$\left.\begin{array}{l} \varphi : U \longrightarrow \varphi(U) \subseteq \hat{\mathbb{R}}^n \\ \\ \varphi : P \longrightarrow \varphi(P) \in \hat{\mathbb{R}}^n \end{array}\right\} \qquad (4.1.1)$$

Since φ is a homeomorphism, its inverse φ^{-1} can be defined as follows :

$$\varphi^{-1} : \varphi(U) \longrightarrow U \qquad (4.1.2)$$

From the first of (4.1.1) and (4.1.2) we conclude that :

$$f \circ \varphi^{-1} : \varphi(U) \longrightarrow \hat{\mathbb{R}}$$

Therefore, $f \circ \varphi^{-1}$ is defined on an open subset of the Cartesian isomanifold $\hat{\mathbb{R}}^n$ and hence we can define its differentialbility. If $f \circ \varphi^{-1}$ is differentiable of class C^k, then f is differentiable of class C^k at the point P.

DEFINITION 4.1.3. Let f be an isofunction which is differentiable of class C^k for every point $P \in M$, then f is called **differentiable of class C^k.**

NOTATION 4.1.1. The set of differentiable isofunctions of class C^k on the differentiable isomanifold M is denoted by :

$$D^0(M)\ ,\ C^k$$

If we consider the set of differential isofunction of class C^∞, then this set is denoted by :

$$D^0(M) \text{ instead of } D^0(M)\ ,\ C^\infty$$

REMARK 4.1.1. The set of all isofunctions on a differentiable isomanifold M of class C^0, are the continuous isofunctions on M. Hence we have :

$$D^0(M), C^0 \subset D^0(M), C^1 \subset \dots \subset D^\infty(M)$$

PROPOSITION 4.1.1. Let M be a differentiable isomanifold. The set of isofunctions of class C^k, $D^0(M), C^k$ can become an isoalgebra over the isofield $\hat{\mathbb{R}}$.

Proof : (i) We consider the first internal decomposition law defined as follows (addition of two isofunction) :

$$f_1 : M \longrightarrow \hat{\mathbb{R}}\ ,\quad f_1 : P \longmapsto f_1(P)$$

$$f_2 : M \longrightarrow \hat{\mathbb{R}}\ ,\quad f_2 : P \longmapsto f_2(P)$$

The addition f_1+f_2 is given by :

$$f_1+f_2 : M \longrightarrow \hat{\mathbb{R}}\ ,\quad (f_1+f_2)(P) \equiv f_1(P)+f_2(P)$$

(ii) The external decomposition law : The product αf_1, where $\alpha \in \hat{\mathbb{R}}$, is defined by :

$$f_1 : M \longrightarrow \hat{\mathbb{R}} \ , \ f_1 : P \longmapsto f_1(P)$$

$$\alpha f_1 : M \longrightarrow \hat{\mathbb{R}} \ , \ \alpha f_1 : P \longmapsto (\alpha f_1)(P) \equiv \alpha f_1(P)$$

(iii) The second internal law of decomposition, product of two functions. It is defined by :

$$f_1 : M \longrightarrow \hat{\mathbb{R}} \ , \ f_1 : P \longmapsto f_1(P)$$

$$f_2 : M \longrightarrow \hat{\mathbb{R}} \ , \ f_2 : P \longmapsto f_2(P)$$

$$f_1 \cdot f_2 : M \longrightarrow \hat{\mathbb{R}}$$

$$f_1 \cdot f_2 : P \longrightarrow (f_1 \cdot f_2)(P) \equiv f_1(P) . f_2(P)$$

These three laws turn out this set into an isoalgebra over the isofield $\hat{\mathbb{R}}$.

DEFINITION 4.1.4 Let M be a differentiable isomanifold with the isoatlas $(U_\alpha, \varphi_\alpha)$ $\alpha \in A$. We consider an open subset V of M and an isofunction :

$$f : M \longrightarrow \hat{\mathbb{R}} \ \text{ such that } \ f(v) \neq 0 \ \forall v \in V$$

If V is the maximal subset M with this property, then the closure $\bar{V}$ of V is called **support** of this function.

THEOREM 4.1.1. Let M be a compact canonical topological space. We consider a family $\{V_\alpha\}$ $\alpha \in A$ of open subsets of M, such that :

$$\bigcup_{\alpha\in A} V_\alpha = M$$

Then, there is a family of isofunctions on M such that each φ_α has a support, contained in V_α with :

$$\varphi_\alpha \geq 0 \qquad \sum_{\alpha\in A} \varphi_\alpha = 1$$

Proof : This theorem, called **partition unity's theorem**, is proved by means of the following remarks :

(i) The isomanifold M can be seperated in subsets, such that each of them is homeomorphic onto an open subset of $\hat{\mathbb{R}}^n$. This means from the topological point of view that the study of each of these subsets is reduced to the open subset of $\hat{\mathbb{R}}^n$.

(ii) From the known theorem for the Cartesian isomanifold $M^n(\hat{\mathbb{R}}) \equiv \hat{\mathbb{R}}^n$ we have :

If W_1 and W_2 are disjoint open subsets of $\hat{\mathbb{R}}^n$, then there exists a function f on $\hat{\mathbb{R}}^n$ with the property :

$$f(w_1)=1 \quad \text{and} \quad f(w_2)=0 \quad \forall w_1\in W_1\ ,\ w_2\in W_2$$

4.2 GERMS OF DIFFERENTIABLE ISOFUNCTIONS.

Let M be a differentiable isofunctions and let P be a point of M. We say that an isofunction f is differentiable near P if f is differentiable on an open neighborhood of P. We shall now introduce an equivalence relation between isofunctions that are differentiable near P. Two isofunctions f_1 and f_2 will be considered equivalent if they have the

same value on a neighborhood of P, each equivalence isofunction being considered at P. Each equivalence class will be called the **germ** of a differentiable isofunction at P. The set of all germs at $P \in M$ will be denoted by F_M. The germ of the isofunction f is denoted by [F].

REMARK 4.2.1 The germ of an isofunction at P depends on the behavior of the isofunction in one open neighborhood of P and not only on the value of the isofunction at P.

All isofunctions of the same germ have the same value at P. This value is called the **value of the germ** at P.

PROPOSITION 4.2.1. The set F_M can be turned out into a commutative algebra.

Proof : Let f_1 and f_2 be two differentiable isofunctions near P and let $[f_1]$ and $[f_2]$ be their germs. On some neighborhood of P the sum f_1+f_2 is defined and it is a differentiable isofunction near P. Now, we can define :

$$[f_1+f_2]$$

to be the germ of isofunction f_1+f_2. Therefore, we have the first internal composition law.

The external composition law : If f is differentiable near P and [f] is its gern, then for every constant $\alpha \in \hat{R}$, αf is differentiable near P and we can define $[\alpha f]$.

These two composition laws are well defined and make F_M. with these three laws becomes a commutative algebra of infinite dimension.

The second internal composition law : If $[f_1]$ and $[f_2]$

are two germs, the second internal composition law, denoted by $\cdot$, is defined as the germ of the product of two representatives $f_1 \cdot f_2$ of $[f_1]$ and $[f_2]$ respectively. It can be easily proved that this product does not depent of the choice of f_1 and f_2.

F_M, with these three laws, bacome a commutative algebra.

REMARK 4.2.1. The germs of differentiable isofunctions on the cartesian isomanifold $M^n(\hat{\mathbb{R}})=\hat{\mathbb{R}}^n$ play important role in many application in different branches of mathematics.

THEOREM 4.2.1. Let M be a differentiable isomanifold. If P and P′ are two points of M, then there is an algebra isomorphism between F_P and $F_{P'}$.

Proof : Let (U,φ) and (U',φ') be two isocharts of M containing the points P and P′ respectively. We can assume $\varphi(P)=\varphi'(P')$. Let $[f]\in F_P$ and let f be the representative of $[f]$. Then f is a differentiable isofunction on some open set V, contained af V. We consider the isofunction :

$$h = f\circ\varphi^{-1}\circ\varphi'$$

which is defined on a neighborhood of P′. Since we have :

$$h\circ\varphi'^{-1} = f\circ\varphi^{-1}$$

we obtain that this isofunction $h\circ\varphi'^{-1}$ is differentiable in a neighborhood of $\varphi'(P')$ and therefore h is differentiable near P′. Let $[h]$ be the germ of h in $F_{P'}$. Now, we construct the mapping t defined by :

$$t : F_P \longrightarrow F_{P'}\ , \quad t : [f] \longrightarrow t([f])=[h]$$

We must show that t is well defined. Let f_1 be another representation of $[f]$. Then $f-f_1\equiv 0$ mod(open) on a neighborhood of P. If we define :

$$h_1 = f_1 \circ \varphi^{-1} \circ \varphi'$$

then $h-h_1=(f-f_1)\circ\varphi^{-1}\circ\varphi'=0$ on a neighborhood of P′ and therefore h and h_1 belong to the same germ of [g].

We can easily prove that the mapping t preserves the three laws in the algebras F_P and $F_{P'}$, that is :

$$t : F_P \longrightarrow F_{P'}\ ,$$
$$t : [f] \longmapsto [h]\ ,$$
$$t : [f'] \longmapsto [h']$$

and t has the properties :

$$t : [f]+[f'] \longmapsto [h]+[h']$$
$$t : \alpha[f] \longmapsto \alpha[h] \quad \forall\alpha\in\hat{\mathbb{R}}$$
$$t : [f]\cdot[f'] \longmapsto [h]\cdot[h']$$

Therefore t is a Lie isomorphism between the two commutive algebra F_P and F'_P.

4.3 ISOVECTOR FIELDS ON AN ISOMANIFOLD

4.3.a Isovector fields on $\hat{\mathbb{R}}^3$

Let Oxyz be a cartesian coordinate system in $\hat{\mathbb{R}}^3$, (figure 1). An isovector field on $\mathbb{R}^3$ is a collection of

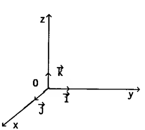


Figure 1

isovector on $\hat{\mathbb{R}}^3$ such that to each point of $\hat{\mathbb{R}}^3$ corresponds one isovector. Therefore an isovector field V on $\hat{\mathbb{R}}^3$ can be written :

$$\vec{V} = \left\{ \vec{V}_P = a_1\vec{I}+a_2\vec{J}+a_3\vec{K} \ / \ P\in\hat{\mathbb{R}}^3,\ a_1,a_2,a_3\in\hat{\mathbb{R}} \right\}$$

and the isovector field $\vec{V}$ on $\hat{\mathbb{R}}^3$ can be written :

$$\vec{V} = f_1(x,y,z)\vec{I} + f_2(x,y,z)\vec{J} + f_3(x,y,z)\vec{K} \qquad (4.3.1)$$

where f_1, f_2 and f_3 are isofunctions on $\hat{\mathbb{R}}^3$. From (4.3.1) we obtain :

$$\vec{V}_P = f_1(p_1,p_2,p_3)\vec{I} + f_2(p_1,p_2,p_3)\vec{J} + f_3(p_1,p_2,p_3)\vec{K} \tag{4.3.2}$$

where (p_1,p_2,p_3) are the isoaffine coordinates of the points P and

$$a_1=f_1(p_1,p_2,p_3)\ ,\quad a_2=f_2(p_1,p_2,p_3)\ ,\quad a_3=f_3(p_1,p_2,p_3)$$

4.3.b Natural basic isovector field on $\hat{\mathbb{R}}^3$. Let $\vec{U}_1$, $\vec{U}_2$ and $\vec{U}_3$ be three isovectors on $\hat{\mathbb{R}}^3$ with the property :

$\vec{U}_1(P) = \vec{I}$, $\vec{U}_2(P) = \vec{J}$ and $\vec{U}_3(P) = \vec{K}$ $\forall P\in\hat{\mathbb{R}}^3$ (figure 2).

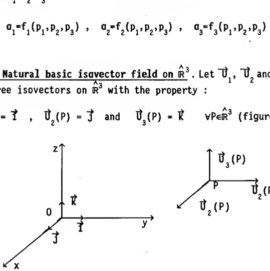


Figure 2

Then $\vec{U}_1$, $\vec{U}_2$ and $\vec{U}_3$ are called **natural basic isovector fields on $\hat{\mathbb{R}}^3$**. For these isovector fields we have :

$$\vec{U}_1 = 1\vec{I}+0\vec{J}+0\vec{K}\ ,\quad \vec{U}_2 = 0\vec{I}+1\vec{J}+0\vec{K}\ ,\quad \vec{U}_3 = 0\vec{I}+0\vec{J}+1\vec{K} \tag{4.3.3}$$

where the quantities 1 and 0 in (4.3.3) are the constants

isofunctions on $\hat{\mathbb{R}}^3$:

$$1 : \hat{\mathbb{R}}^3 \longrightarrow \hat{\mathbb{R}} \ , \quad 1 : P \longmapsto 1(P)=1 \quad \forall P \in \hat{\mathbb{R}}^3$$

$$0 : \hat{\mathbb{R}}^3 \longrightarrow \hat{\mathbb{R}} \ , \quad 0 : P \longmapsto 0(P)=0 \quad \forall P \in \hat{\mathbb{R}}^3$$

NEW NOTATION 4.3.1. Let $\vec{V}$ be an isovector field on $\hat{\mathbb{R}}^3$. This can be written :

$$\vec{V} = f_1(x,y,z)\vec{I}+f_2(x,y,z)\vec{J}+f_3(x,y,z)\vec{K} \qquad (4.3.4)$$

and if we take under consideration the natural basic isovector fields, then (4.3.4) takes the form :

$$\vec{V} = f_1(x,y,z)\vec{U}_1 + f_2(x,y,z)\vec{U}_2 + f_3(x,y,z)\vec{U}_3 \quad (4.3.5)$$

In some cases the natural basic isovector fields can be represented by the symbols :

$$\vec{U}_1 = \frac{\partial}{\partial x} \quad , \quad \vec{U}_2 = \frac{\partial}{\partial y} \quad , \quad \vec{U}_3 = \frac{\partial}{\partial z} \quad (4.3.6)$$

Therefore, the isovectors field $\vec{V}$ defined by (4.3.5) by means of (4.3.6) can take the form :

$$\vec{V} = f_1(x,y,z)\frac{\partial}{\partial x} + f_2(x,y,z)\frac{\partial}{\partial y} + f_3(x,y,z)\frac{\partial}{\partial z} \qquad (4.3.7)$$

DEFINITION 4.3.1 The isovector field $\vec{V}$ is called of class C^k if the isofunctions f_1, f_2 and f_3 are of class C^k, that means f_1, f_2 and f_3 are differentiable k times.

NOTATION 4.3.2. The set of all isovector fields of cless C^k is denoted by :

$$D^1(\hat{\mathbb{R}}^3)\ ,\ C^k$$

If $k=\infty$, then the set $D^1(\hat{\mathbb{R}}^3)$ C^∞ is denoted by $D^1(\hat{\mathbb{R}}^3)$

PROPOSITION 4.3.1 Let $\vec{V} = f_1\frac{\partial}{\partial x} + f_2\frac{\partial}{\partial y} + f_3\frac{\partial}{\partial z}$ be an isovector field. Then $\vec{V}$ can be considered as a derivation on the algebra $D^0(\hat{\mathbb{R}}^3)$, C^∞ or briefly on $D^0(\hat{\mathbb{R}}^3)$.

Proof : Firstly, we prove that $\vec{V}$ can be a mapping on $D^0(\hat{\mathbb{R}}^3)$, as follows :

$$V : D^0(\hat{\mathbb{R}}^3) \longrightarrow D^0(\hat{\mathbb{R}}^3)$$

$$\vec{V} : f = f_1\frac{\partial}{\partial x} + f_2\frac{\partial}{\partial y} + f_3\frac{\partial}{\partial z} \longmapsto$$

$$\vec{V}(f) = f_1\frac{\partial f}{\partial x} + f_2\frac{\partial f}{\partial y} + f_3\frac{\partial f}{\partial z}$$

This mapping $\vec{V}$ has the property :

$$\vec{V} : a_1\varphi_1+a_2\varphi_2 \longmapsto \vec{V}(a_1\varphi_1+a_2\varphi_2) = f_1\frac{\partial}{\partial x}(a_1\varphi_1+a_2\varphi_2) +$$

$$+ f_2\frac{\partial}{\partial y}(a_1\varphi_1+a_2\varphi_2) + f_3\frac{\partial}{\partial z}(a_1\varphi_1+a_2\varphi_2) = a_1 f_1\frac{\partial\varphi_1}{\partial x} +$$

$$+ a_2 f_1\frac{\partial\varphi_2}{\partial x} + a_1 f_2\frac{\partial\varphi_1}{\partial x} + a_2 f_2\frac{\partial\varphi_2}{\partial x} + a_1 f_3\frac{\partial\varphi_1}{\partial x} +$$

$$+ a_2 f_3\frac{\partial\varphi_2}{\partial x} = a_1\left(f_1\frac{\partial\varphi_1}{\partial x} + f_2\frac{\partial\varphi_1}{\partial y} + f_3\frac{\partial\varphi_1}{\partial z}\right) +$$

$$+ \ o_2\left(f_1 \frac{\partial \varphi_2}{\partial x} + f_2 \frac{\partial \varphi_2}{\partial y} + f_3 \frac{\partial \varphi_2}{\partial z}\right) = o_1\vec{V}(\varphi_1) + o_2\vec{V}(\varphi_2) \tag{4.3.8}$$

where o_1, $o_2 \in \hat{\mathbb{R}}$ and φ_1, $\varphi_2 \in D^0(\hat{\mathbb{R}}^3)$, $\mathcal{C}^k$.

The relation (4.3.8) shows that the mapping $\vec{V}$ is linear on the vector space $D^0(\hat{\mathbb{R}}^3)$. For the linear mapping $\vec{V}$ we have :

$$\vec{V} : \varphi_1\varphi_2 \longrightarrow \vec{V}(\varphi_1\varphi_2) = f_1 \frac{\partial}{\partial x} \varphi_1\varphi_2 + f_2 \frac{\partial}{\partial y} \varphi_1\varphi_2 +$$

$$f_3 \frac{\partial}{\partial z} \varphi_1\varphi_2 = f_1(\varphi_2 \frac{\partial}{\partial x} \varphi_1 + \varphi_1 \frac{\partial}{\partial x} \varphi_2) +$$

$$+ f_2(\varphi_2 \frac{\partial}{\partial y} \varphi_1 + \varphi_1 \frac{\partial}{\partial y} \varphi_2) + f_3(\varphi_2 \frac{\partial}{\partial z} \varphi_1 + \varphi_1 \frac{\partial}{\partial z} \varphi_2) =$$

$$= \varphi_2(f_1 \frac{\partial}{\partial x} \varphi_1 + f_2 \frac{\partial}{\partial y} \varphi_1 + f_3 \frac{\partial}{\partial z} \varphi_1) +$$

$$+ \varphi_1(f_1 \frac{\partial}{\partial x} \varphi_2 + f_2 \frac{\partial}{\partial y} \varphi_2 + f_3 \frac{\partial}{\partial z} \varphi_2) = \varphi_2\vec{V}(\varphi_1) + \varphi_1\vec{V}(\varphi_2) \tag{4.3.9}$$

Relation (4.3.9) proves that the linear mapping $\vec{V}$ is a derivation on the algebra $D^0(\hat{\mathbb{R}}^3)$.

PROPOSITION 4.3.3 Every isovector field :

$$\vec{V} = f_1 \frac{\partial}{\partial x} + f_2 \frac{\partial}{\partial y} + f_3 \frac{\partial}{\partial z}$$

can be considered as a linear mapping of the isovector space $D^0(\hat{\mathbb{R}}^3)$, $\mathcal{C}^k$ into the isovector space $D^0(\hat{\mathbb{R}}^3)$, $\mathcal{C}^{k-1}$ k>0.

Proof : The mapping, which can be determined by the $\vec{\nabla}$, denoted also by $\vec{\nabla}$, is defined as follows :

$$\vec{\nabla} : D^0(\hat{\mathbb{R}}^3),\ \mathcal{C}^k \longrightarrow D^0(\hat{\mathbb{R}}^3),\ \mathcal{C}^{k-1}$$

$$\vec{\nabla} : \varphi = f_1\frac{\partial}{\partial x} + f_2\frac{\partial}{\partial y} + f_3\frac{\partial}{\partial z} \ :\longmapsto \vec{\nabla}(\varphi) =$$

$$= f_1\frac{\partial\varphi}{\partial x} + f_2\frac{\partial\varphi}{\partial y} + f_3\frac{\partial\varphi}{\partial z}$$

For this mapping we have :

$$\vec{\nabla} : a_1\varphi_1+a_2\varphi_2 \longmapsto \vec{\nabla}(a_1\varphi_1+a_2\varphi_2) = f_1\frac{\partial}{\partial x}(a_1\varphi_1+a_2\varphi_2) +$$

$$+ f_2\frac{\partial}{\partial y}(a_1\varphi_1+a_2\varphi_2) + f_3\frac{\partial}{\partial z}(a_1\varphi_1+a_2\varphi_2) = a_1f_1\frac{\partial}{\partial x}\varphi_1 +$$

$$+ a_2f_1\frac{\partial}{\partial x}\varphi_2 + a_1f_2\frac{\partial}{\partial y}\varphi_1 + a_2f_2\frac{\partial}{\partial y}\varphi_2 + a_1f_3\frac{\partial}{\partial z}\varphi_1 +$$

$$+ a_2f_3\frac{\partial}{\partial z}\varphi_2 = a_1\Big(f_1\frac{\partial}{\partial x}\varphi_1 + f_2\frac{\partial}{\partial y}\varphi_1 + f_3\frac{\partial}{\partial z}\varphi_1\Big) +$$

$$+ a_2\Big(f_1\frac{\partial}{\partial x}\varphi_2 + f_2\frac{\partial}{\partial y}\varphi_2 + f_3\frac{\partial}{\partial z}\varphi_2\Big) = a_1\vec{\nabla}(\varphi_1) + a_2\vec{\nabla}(\varphi_2) \tag{4.3.10}$$

where $a_1,\ a_2\in\hat{\mathbb{R}}\ ,\ \varphi_1,\varphi_2\in D^0(\hat{\mathbb{R}}^3),\ \mathcal{C}^k$.

The relation (4.3.10) shows that the mapping $\vec{\nabla}$ is linear between the vector spaces $D^0(\hat{\mathbb{R}}^3)$, $\mathcal{C}^k$ and $D^0(\hat{\mathbb{R}}^3)$, $\mathcal{C}^{k-1}$.

4.3.c Isovector fields on $\hat{\mathbb{R}}^n$. All notions, propositions and remarks for the isovector fields on $\hat{\mathbb{R}}^3$ can be extended on $\hat{\mathbb{R}}^n$. Therefore, every isovector field $\vec{V}$ on $\hat{\mathbb{R}}^n$ take the form :

$$\vec{V} = f_1(x_1,\dots,x_n)e_1+\dots+ f_n(x_1,\dots,x_n)e_n$$

where $f_1(x_1,\dots,x_n),\dots,f_n(x_1,\dots,x_n)$ are isofunctions on $\hat{\mathbb{R}}^n$ and $e_1,\dots,e_n$ are the canonical base of the Cartesian isovector space $\vec{V}(\hat{\mathbb{R}})$. The isovector field $\vec{V}$ can be also written :

$$\vec{V} = f_1(x_1,\dots,x_n)\frac{\partial}{\partial x_1}+\dots+f_n(x_1,\dots,x_n)\frac{\partial}{\partial x_n}$$

PROPOSITION 4.3.4 Every isovector field on $\hat{\mathbb{R}}^n$ can be considered as a derivation on the algebra $D^0(\hat{\mathbb{R}})$.

Proof : This is similar as the proposition 4.3.1 $\vec{V}$, as mapping, is defined by :

$$\vec{V} : \varphi \longmapsto \vec{V}(\varphi) = \sum_{k=1}^{n} f_k \frac{\partial\varphi}{\partial x_k} = f_1\frac{\partial\varphi}{\partial x_1}+\dots+f_n\frac{\partial\varphi}{\partial x_n}$$

4.3.3 New notation. The isovector field on $\hat{\mathbb{R}}^n$ are denoted usually without arrow, that is :

$$V = f_1\frac{\partial}{\partial x_1}+\dots+f_n\frac{\partial^n}{\partial x_n} = \sum_{i=1} f_i\frac{\partial}{\partial x_i}$$

or

$$V = f_1e_1+\dots+f_ne_n = \sum_{i=1}^{n} f_ie_i$$

IMPORTANT REMARK 4.3.1. The consideration of an isovector field on $\hat{\mathbb{R}}^n$ as a derivation on the algebra $D^0(\mathbb{R}^n)$ permits the definition of an isovector field on a differentiable isomanifold.

4.3.d. Isovector feilds on isomanifolds. Let M be a differentiable isomanifold. Let $D^0(M)$ be the algebra of all isofunctions on M. Every derivation on $D^0(M)$ is called **isovector field** on the differentiable isomanifold. If X is an isovector field on M, then X has the properties :

$$X : D^0(M) \longrightarrow D^0(M) \quad , \quad X : f \longmapsto X(f)$$

$$X : a_1f_1+a_2f_2 \longmapsto X(a_1f_1+a_2f_2) = a_1X(f_1)+a_2X(f_2)$$

$$X : f_1f_2 \longmapsto X(f_1f_2) = f_1X(f_2)+f_2X(f_1)$$

where a_1, $a_2\in\hat{\mathbb{R}}$ and f_1, $f_2\in D^0(M)$.

PROPOSITION 4.3.5. Let M be a differentiable isomanifold. The set of isovector fields on M, denoted by $D^1(M)$, can be turned into a Lie algebra over $D^0(M)$.

Proof : On the set $D^1(M)$ we define the following operations :
(i) The internal composition law : If $X,Y\in D^1(M)$, then we have :

$$X : D^0(M) \longrightarrow D^0(M) \quad , \quad X : f \longmapsto X(f)$$

$$Y : D^0(M) \longrightarrow D^0(M) \quad , \quad Y : f \longmapsto Y(f)$$

Now, we define the sum of X+Y as follows :

$$X+Y : D^1(M) \longrightarrow D^0(M)\ ,\ X+Y : f \longmapsto (X+Y) \equiv X(f)+Y(f)$$

(ii) <u>The external composition law</u> : If $X \in D^0(M)$ and $\varphi \in D^0(M)$, then φX is defined by :

$$X : D^0(M) \longrightarrow D^0(M)\ ,\ X : f \longmapsto X(f)$$

$$\varphi X : D^0(M) \longrightarrow D^0(M)\ ,\ \varphi X : f \longmapsto (\varphi X)(f) \equiv \varphi X(f)$$

These two laws have the properties, which turn $D^1(M)$ into a module over $D^0(M)$.

(iii) <u>Second internal composition law</u>. On $D^1(M)$ we define the second internal composition law, denoted by [ ], as follows

$$[\ ] : D^1(M) x D^1(M) \longrightarrow D^1(M)$$

$$[\ ] : (X,Y) \longmapsto [X \hat{,} Y] \equiv X{*}Y - Y{*}X = XTY - YTX$$

where T is the inverse of the isounit $\hat{I}$ of the underlying isofield $\hat{R}$, by reaching in this was the fundamental isotopy of Lie algebra by Santilli (1978a). Note that the composition $D^1(M) x D^1(M)$ is among isospaces and, as such, isotopic, i.e., realized by $X{*}Y - Y{*}X$.

It can be easily proved that the Lie-Santilli bracket satisfies the relations :

$$[X \hat{,} Y] = -[Y \hat{,} X]$$

$$[X \hat{,} [Y \hat{,} Z]] + [Y \hat{,} [Z \hat{,} X]] + [Z \hat{,} [X \hat{,} Y]] = 0$$

Hence $D^1(M)$ become a Lie algebra over $D^0(M)$.

REMARK 4.3.2. For each element of $D^1(M)$, which is an isovector field, we have assumed that it is a derivation on the algebra $D^0(M)$. With the same manner we obtain the set $D^1(M)$, $\mathcal{C}^k$. Each element of $D^1(M)$, $\mathcal{C}^k$ is called **isovector field** of class $\mathcal{C}^k$. This can be considered as a derivation on the algebra $D^1(M)$, $\mathcal{C}^k$.

4.3.e Isovector fields on an isochart of an isomanifold.

Let M be a differentiable isomanifold and (U,φ) an isochart of M. Therefore we have :

$$\varphi : U \longrightarrow \varphi(U) \subseteq \hat{\mathbb{R}}^n \quad ,$$

$$\varphi : q \longmapsto \varphi(q)=(x_1(q),\ldots,x_n(q)) \qquad \forall q \in U$$

where n is the dimension of M and $(x_1,\ldots,x_n)$ are the isocoordinates of M on its open neighborhood U. Let f be an isofunction on M, that is :

$$f : M \longrightarrow \mathbb{R} \ , \quad \hat{f}_{/U} : U \longrightarrow \hat{\mathbb{R}} \qquad (4.3.11)$$

Then for φ^{-1} we have :

$$\varphi^{-1} : \varphi(U) \longrightarrow U$$

From the composition of f and φ^{-1} we obtain :

$$f_{/U} \circ \varphi^{-1} : f(U) \longrightarrow \hat{\mathbb{R}}$$

Hence the isofunction $f_{/U} \circ \varphi^{-1}$ is a function on $\varphi(U)$, which is denoted by f^*.

If we consider the restriction of the isofunctions $D^0(M)$ on U, we obtain the set $D^0(M)$, that is, the isofunctions on U. It can be easily proved that the set $D^0(U)$ can be become an isoalgebra over $\hat{\mathbb{R}}$.

We consider the derivations $I_1, \ldots, I_n$ on $D^1(U)$, which are defined by :

$$I_1 : D^1(U) \longrightarrow D^1(U) \quad I_1 : f \longmapsto I_1(f) = \frac{\partial f^*}{\partial x_1} \circ \varphi$$

....

$$I_n : D^1(U) \longrightarrow D^1(U) \quad I_n : f \longmapsto I_n(f) = \frac{\partial f^*}{\partial x_n} \circ \varphi$$

It can be easily proved that the derivations $I_1, \ldots, I_n$, which are isovector fields on U, form a base, denoted with :

$$I_1 = \frac{\partial}{\partial x_1}, \ldots, I_n = \frac{\partial}{\partial x_n}$$

Hence, every vector field X on U can be written :

$$X = f_1 \frac{\partial}{\partial x_1} + \ldots + f_n \frac{\partial}{\partial x_n}$$

where $f_1, \ldots, f_n \in D^0(U)$

REMARK 4.3.3. The isovector fields on an isochart (U,φ) of an isomanifold behave as isovector fields on an open set of $\hat{\mathbb{R}}^n$.

4.4 ISOTANGENT SPACE AT A POINT OF AN ISOMANIFOLD

DEFINITION 4.4.1. Let P be a point of the differentiable isomanifold M. Let (U,φ) be an isochart arount the point P. Therefore, we have :

$$\varphi : U \longrightarrow \varphi(U) \subset \hat{\mathbb{R}}^n$$

If f is an isofunction on M, then we obtain the isofunction

$$f^* = f \circ \varphi^{-1} : \varphi(U) \longrightarrow \hat{\mathbb{R}}$$

which has the form :

$$f^*(x_1, \ldots, x_n)$$

where $x_1, \ldots, x_n$ the isocoordinates on U. We consider the mapping :

$$I : D^0(U) \longrightarrow D^0(U) \quad , \quad I : f \longrightarrow I(f) = f^* \circ \varphi$$

The fact that $f^* \circ \varphi$ is an isofunction on U can be proved via

the property :

$$\varphi : U \longrightarrow \varphi(U) \quad , \quad f^{*} : \varphi(U) \longrightarrow \hat{\mathbb{R}}$$

whose composition implies :

$$f^{*}\circ\varphi : U \longrightarrow \hat{\mathbb{R}}$$

It can be easily proved that I is a derivation on $\overset{0}{D}(U)$. Therefore, I is an isovector field on U, which sometimes will be denoted by X. From X and for the point P we can construct the following linear mapping :

$$X_{p} : D^{0}(\hat{\mathbb{R}}) \longrightarrow \hat{\mathbb{R}}$$

$$X_{p} : f \longmapsto X_{p}(f)=f^{*}\circ\varphi(P) = (X(f))_{p}$$

We construct the set $T_{p}(M)$ defined by :

$$T_{p}(M) = \left\{ X_{p} \ / \ X\in D^{1}(M) \ , \ X_{p} : D^{0}(M) \longrightarrow \mathbb{R} \ , \ \text{lin. mapping}\right\}$$

On this set we define the following composition laws :

(i) Internal composition law : If X_{p}, $Y_{p}\in T_{p}(M)$, then $X_{p}+Y_{p}$ is defined as follows :

$$X_{p} : D^{0}(M) \longrightarrow \hat{\mathbb{R}} \quad , \quad X_{p} : f \longmapsto (X(f))_{p}$$

$$Y_{p} : D^{0}(M) \longrightarrow \hat{\mathbb{R}} \quad , \quad Y_{p} : f \longmapsto (Y(f))_{p}$$

The sum $X_{p}+Y_{p}$ is defined by :

$$X_{p}+Y_{p} : D^{0}(M) \longrightarrow \hat{\mathbb{R}}$$

$$X_P : f \longmapsto (X_P+Y_P)(f) \equiv (Xf)_P+(Yf)_P = \{(X+Y)(f)\}_P$$

(ii) External composition law : If $\alpha\in\hat{\mathbb{R}}$, then αX_P is defined as follows :

$$\alpha X_P : D^0(M) \longrightarrow \hat{\mathbb{R}}$$

$$\alpha X_P : f \longmapsto (\alpha X_P)(f) \equiv (\alpha X(f))_P$$

This set $T_P(M)$ with these two laws becomes an isovector space over $\hat{\mathbb{R}}$ of dimension n and it is called **isotangent space** of M at the point P.

4.4.a A base of the isotangent space $T_P(M)$. Let (U,φ) be an isochart of the differentiable isomanifold. Let $(x_1,\dots,x_n)$ be an isocoordinate system on U. It has been proved that the isovector fields :

$$I_1 = \frac{\partial}{\partial x_1}, \dots, I_n = \frac{\partial}{\partial x_n}$$

form a base of $D^1(U)$ with coefficients from $D^1(U)$. Let $T_P(M)$ be the isotangent space of M at the point $P\in U$.

The values of the isovector fields at the point P are given by :

$$e_1=(I_1)_P=\left(\frac{\partial}{\partial x_1}\right)_P, \dots, e_n=(I_n)_P=\left(\frac{\partial}{\partial x_n}\right)_P$$

and they are isovectors belonging to the isotangent space $T_P(M)$. The isovectors $e_1,\dots,e_n$ form a base of $T_P(M)$. For these isovectors e_i, $i=1,\dots,n$, we have the mappings :

$$e_i : D^0(M) \longrightarrow D^0(M) \ , \quad e_i : f \longmapsto e_i(f) = \frac{\partial f^*}{\partial x_i} \circ \varphi$$

where f^* has been defined above.

DEFINITION 4.4.2. Let M be a differentiable isomanifold of dimension 2, which is called **isosurface**. The isotangent space of M at the point P is called **isotangent plane** at the point P and denoted also by $T_P(M)$.

DEFINITION 4.4.3. Let c be a differentiable isomanifold of dimension 1, which is called **isocurve**. The isotangent space of c at its point P is called **isotangent** and denoted by $\dot{c}_P$. $\dot{c}_P$ is an isovector space of one dimension, that is $\dot{c}_P \cong \hat{\mathbb{R}}$.

REMARK 4.4.1. If the isomanifold M is of zero dimension, i.e., it consists of isolated points, then the isotangent space at each of these points coincide with the same point.

PROPOSITION 4.4.1 Let M be a differentiable isomanifold of dimension n, whose one isoatlas $(U_\alpha, \varphi_\alpha)$ $\alpha \in A$ consists of at least two isocharts. If $P \in U_\alpha \cap U_\beta \neq \emptyset$, then determine the relation between the two bases :

$$\left\{ \left(\frac{\partial}{\partial x_1} \right)_P , \ldots , \left(\frac{\partial}{\partial x_n} \right)_P \right\} \ , \quad \left\{ \left(\frac{\partial}{\partial y_1} \right)_P , \ldots , \left(\frac{\partial}{\partial y_n} \right)_P \right\} \tag{4.4.1}$$

On $T_P(M)$, where $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are the

isocoordinates on the isocharts $(U_\alpha,\varphi_\alpha)$ and (U_β,φ_β), respectively.

Proof : The isocoordinates $(x_1,\ldots,x_n)$ and $(y_1,\ldots,y_n)$ on the isocharts $(U_\alpha,\varphi_\alpha)$ and (U_β,φ_β) respectively are connected on $U_\alpha \cap U_\beta$ by the relations :

$$y_i = \lambda_i(x_1,\ldots,x_n) \quad i=1,\ldots,n \tag{4.4.2}$$

The two bases (4.4.1) are related by :

$$\begin{aligned}
\left(\frac{\partial}{\partial y_1}\right)_P &= \frac{\partial \lambda_1}{\partial x_1}\left(\frac{\partial}{\partial x_1}\right)_P + \ldots + \frac{\partial \lambda_1}{\partial x_n}\left(\frac{\partial}{\partial x_n}\right)_P \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\left(\frac{\partial}{\partial y_n}\right)_P &= \frac{\partial \lambda_n}{\partial x_1}\left(\frac{\partial}{\partial x_1}\right)_P + \ldots + \frac{\partial \lambda_n}{\partial x_n}\left(\frac{\partial}{\partial x_n}\right)_P
\end{aligned} \tag{4.4.3}$$

From (4.4.3) we obtain the matrix :

$$A = \begin{pmatrix} \frac{\partial \lambda_1}{\partial x_1} & \cdots\cdots\cdots & \frac{\partial \lambda_1}{\partial x_n} \\ & \cdots\cdots\cdots\cdots\cdots & \\ \frac{\partial \lambda_n}{\partial x_1} & & \frac{\partial \lambda_n}{\partial x_n} \end{pmatrix}_P = \left(\frac{D(\lambda_1,\ldots,\lambda_n)}{D(x_1,\ldots,x_n)}\right)_P \tag{4.4.4}$$

Therefore, the matrix, which corresponds to the change of the

two bases is the transpose of the matrix A, that is :

$$t_A = B = \begin{pmatrix} \frac{\partial\lambda_1}{\partial x_1} & \cdots\cdots\cdots & \frac{\partial\lambda_n}{\partial x_1} \\ \cdots & \cdots\cdots\cdots & \cdots \\ \frac{\partial\lambda_1}{\partial x_n} & & \frac{\partial\lambda_n}{\partial x_n} \end{pmatrix}_P$$

which is the transpose of the Jacobian matrix (4.4.4) of (4.4.2).

DEFINITION 4.4.4 Let P be a point of the differentiable isomanifold M. The isotangent space $T_P(M)$ of M at P is an isovector space of dimansion n. Therefore $T_P(M)$ has a dual space, which is an isovector space of dimension n and is called **isocotangent** space.

4.5 DIFFFERENTIABLE ISOEXTERIOR FORMS OF THE FIRST ORDER

DEFINITION 4.5.1 Let M be a differentiable isomanifold of dimension n. From M we obtain the isoalgebra $D^0(M)$ and the isomodule $D^1(M)$ over $D^0(M)$. We denote by $D_1(M)=D^1(M)^*$ the isodual of $D^1(M)$. Therefore $D_1(M)$ is defined by :

$$D_1(M) = \left\{ w \ / \ w : D^1(M) \longrightarrow D^0(M) \ , \ w\text{=linear isoform} \right\}$$

Each w is called **differentiable isoexterior form of the first order** or **differentiable isoexterior 1-form on M**, as introduced for the first time by Santilli (1991b, Memoir 25)

PROPOSITION 4.5.1 The set $D_1(M)$ can be become an isomodule over $D^0(M)$.

Proof : We define on $D_1(M)$ the following laws :
(i) Internal composition law : If

$$w_1 : D^1(M) \longrightarrow D^0(M) \quad , \quad w_1 : X \longmapsto w_1(X)$$

$$w_2 : D^1(M) \longrightarrow D^0(M) \quad , \quad w_2 : X \longmapsto w_2(X)$$

The summation w_1+w_2 is as follows :

$$w_1+w_2 : D^1(M) \longrightarrow D^0(M)$$

$$w_1+w_2 : X \longmapsto (w_1+w_2)(X) \equiv w_1(X)+w_2(X)$$

(ii) External composition law : If $f \in D^0(M)$, then the product fw_1 is defined by :

$$fw_1 : D^1(M) \longrightarrow D^0(M) \quad , \quad fw_1 : X \longmapsto (fw_1)(X) \equiv fw_1(X)$$

This set $D_1(M)$ with these two laws become an isomodule over the isoalgebra $D^0(M)$. The elements of $D_1(M)$, as it is known, are called differential isoexterior forms of the first order

or differentiable isoexterior 1-form. There on, when we write isoexterior 1-forms we mean differentiable.

4.5.1 Isoexterior forms of the first order and differentiability of class $\mathcal{C}^k$.

If we use the isoalgebra $D^0(M)$, $\mathcal{C}^k$ instead of $D^0(M)$ and the isomodule $D^1(M)$, $\mathcal{C}^k$ instead of $D^1(M)$, then we obtain the isoexterior form of the first order and differiability of class $\mathcal{C}^k$, denoted by $D_1(M)$, $\mathcal{C}^k$ and defined by :

$$D_1(M),\ \mathcal{C}^k = \left\{ w \ / \ w{=}D^1(M),\mathcal{C}^k \longrightarrow D^0(M),\mathcal{C}^k,\ w{=}\text{isolin. form} \right\}$$

It can be easily proved that $D_1(M),\mathcal{C}^k$ is an isomodule over the isoalgebra $D^0(M),\mathcal{C}^k$

4.5.2 Isoexterior forms on an isochart.

Let (U,φ) be an isochart on the differentiable isomanifold M. Let $(x_1,\dots,x_n)$ be the local isocoordinates on U. From U we obtain the isoalgebra $D^0(U)$ and the isomodule $D^1(U)$ of isovector fields on U, which can be an isomodule over $D^0(U)$. Let $D^1(U)^*$ be the dual of $D^1(U)$ denoted by $D_1(U)$. Each element of $D_1(U)$ is called **differentiable isoexterior form of the first order on U**, or **differentiable isoexterior 1-form.**

It can be easily proved that $D_1(U)$ can be an isomodule

over the isoalgebra $D^0(U)$.

We have shown that the isovector fields :

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$$

form a base of the isomodule $D^1(U)$ of dimension n over the isoalgebra $D^0(U)$. If $X \in D^1(U)$, then we have :

$$X = f_1 \frac{\partial}{\partial x_1} + \ldots + f_n \frac{\partial}{\partial x_n}$$

The dual base of $\left\{ \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right\}$ is denoted by :

$$\left\{ dx_1, \ldots, dx_n \right\}$$

and is a base of $D_1(U)$. The two bases are related by the relations :

$$(dx_i)\left(\frac{\partial}{\partial x_j} \right) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

The numbers δ_{ij} are called Kronecker's symbols.

If $w \in D_1(M)$ a differentiable isoexterior 1-form, then this can written (Santilli 1991b) :

$$w = \varphi_1 dx_1 + \ldots + \varphi_n dx_n$$

where $\varphi_1, \ldots, \varphi_n \in D^0(U)$, or more explicity,

$$w = \sum_{i,j} \varphi_i^0 T_{ij} dx_j$$

where $T=(T_{ij})$ is the inverse of the underlying isounit $\hat{I}$ and φ^0's are ordinary functions.

If $X \in D^1(M)$, then we have :

$$X = f_1 \frac{\partial}{\partial x_1} + \ldots + f_n \frac{\partial}{\partial x_n}$$

where $f_1, \ldots, f_n \in D^0(U)$ and

$$w(X) = f_1\varphi_1 + \ldots + f_n\varphi_n$$

4.5.3 __A base of the isotangent space.__ Let P be a point of the isochart (U,φ) of the isomanifold M. Let $(x_1,\ldots,x_n)$ be local isocoordinates on U. Let $T_P(M)$ and $T_P(M)$ be the tangent and cotangent of M at P. It is known that the isovector fields :

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$$

form a base of $D^1(U)$ and the isoexterior 1-forms :

$$dx_1, \ldots, dx_n$$

form a base of $D_1(U)$. The isovectors :

$$\left(\frac{\partial}{\partial x_1}\right)_P, \ldots, \left(\frac{\partial}{\partial x_n}\right) \qquad (4.5.2)$$

form a base of $T_p(M)$ and

$$(dx_1)_p, \ldots, (dx_n)_p$$

form a base of T_p^* which is dual of (4.5.2). Therefore, we have :

$$(dx_i)_p\left(\frac{\partial}{\partial x_j}\right)_p = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i\neq j \end{cases}$$

4.6 ISOTENSOR FIELDS

DEFINITION 4.6.1 Let M be a differentiable isomanifold of dimension n. From this we obtain the isoalgebra $D^0(M)$ of all isofunctions on M and the isomodules $D^1(M)$ and $D_1(M)$ of the isovector fields and isoexterior 1-form on M, respectively.

We consider the Cartesian product :

$$\underbrace{D^1 x D^1 x \ldots x D^1}_{s \text{ times}}$$

that is, D^1 is applied s-times. We construct the set :

$$T = \left\{ f : \underbrace{D^1 x D^1 x \ldots x D^1}_{s \text{ times}} \longrightarrow D^0 \text{ , } f = s\text{-multilinear form} \right\}$$

that is, T is the set of all s-multilinear forms on T. Each f is called differentiable covariant isotensor field of

orders s, or simply covariant isotensor field of order s, first introduced in Santilli (1991b).

PROPOSITION 4.6.I. The set T can be an isomodule on the oalgebra $D^0(M)$.

Proof : On the set T we define the following composition laws :

(i) Internal composition law :

$$f_1 : D^1 x \ldots x D^1 \longrightarrow D^0 ,$$

$$f_1 : (x_1, \ldots, x_s) \longmapsto f_1(x_1, \ldots, x_s)$$

$$f_2 : D^1 x \ldots x D^1 \longrightarrow D^0 ,$$

$$f_2 : (x_1, \ldots, x_s) \longmapsto f_2(x_1, \ldots, x_s)$$

The sum f_1+f_2 is defined by :

$$f_1+f_2 : D^1 x \ldots x D^1 \longrightarrow D^0 ,$$

$$f_1+f_2 : (x_1, \ldots, x_s) \longmapsto (f_1+f_2)(x_1, \ldots, x_s) \equiv$$

$$\equiv f_1(x_1, \ldots, x_s)+f_2(x_1, \ldots, x_s)$$

(ii) The external composition law : If $\varphi \in D^0$, then the multiplication φf_1 is defined by :

$$\phi f_1 : D^1 x \ldots x D^1 \longrightarrow D^0 ,$$

$$\phi f_1 : (x_1, \ldots, x_n) \longmapsto (\phi f_1)(x_1, \ldots, x_n) \equiv$$

$$\equiv \phi f_1(x_1, \ldots, x_s) \text{ multiplication of functions.}$$

The set T denoted by :

$$T = \hat{\otimes} D_1$$

becomes an isomodule over $D^0(M)$ of dimension n^s.

DEFINITION 4.6.2 Let $D^0(M)$, $D^1(M)$ and $D_1(M)$ be the isoalgebra and the isomodules of the differentiable isomanifold defined above. We consider the Cartesian product :

$$\underbrace{D_1 x D_1 x \ldots x D_1}_{r \text{ times}}$$

that is, D_1 is applied r-times. We construct the set :

$$\Sigma = \left\{ \psi \;/\; \psi : D_1 x D_1 x \ldots x D_1 \longrightarrow D^0 \text{ r-multilinear form} \right\}$$

that is, ψ is an r-multilinear form on D^0. Each ψ is called differentiable contravariant isotensor field of order r or briefly contravariant isotensor field of order r.

PROPOSITION 4.6.2. The set Σ of all contravariant isotensor field of order r denoted by :

$$\Sigma = \hat{\otimes} D^1$$

can be an isomodule of dimension n^r over the isoalgebra

$D^0(M)$.

Proof : This propertycan be proved with the same manner as proposition 4.6.1.

DEFINITION 4.6.3 Let M be a differentiable isomanifold. From this we obtain the isomodules $D^1(M)$ and $D_1(M)$ over the isoalgebra $D^0(M)$.

We consider the Cartesian product :

$$\underbrace{D^1 x \ldots x D^1}_{\text{s times}} x \underbrace{D_1 x \ldots x D_1}_{\text{r-times}}$$

that means the Cartesian product of D^1 r-times and of D_1 s-times. We construct the set :

$$V = \left\{ \varphi / \varphi : D^1 x \ldots x D^1 x D_1 x \ldots x D_1 \longrightarrow D^0 \text{ , } \varphi = (r+s)\text{-multilinear form} \right\}$$

Each element φ of V is called differentiable isotensor field of type (r,s) or briefly isotensor field of type (r,s). This set is denoted by :

$$V = \overset{r}{\otimes} D^1 \overset{s}{\otimes} D_1$$

PROPOSITION 4.6.3. The set $\overset{r}{\otimes} D^1 \overset{s}{\otimes} D_1$ of all isotensor fields of type (r,s) can be become an isomodule over $D^0(M)$ of dimension n^{r+s}.

Proof : On the set $\overset{r}{\otimes} D^1 \overset{s}{\otimes} D_1$ we define the same composition

laws, as in proposition 4.6.1 which turns this set into an isomodule over $D^0(M)$.

4.6.4 **Isotensor fields on an isochart.** Let (U,φ) be an isochart of the differentiable isomanifold M. In a similar manner, as above, we can construct the isomodules :

$$\overset{r}{\otimes}D^1(U) \quad , \quad \overset{s}{\otimes}D_1(U) \quad , \quad \overset{r}{\otimes}D^1(U) \quad , \quad \overset{s}{\otimes}D_1(U)$$

over the isoalgebra $D^0(U)$, whose elements :

$$\alpha\in\overset{r}{\otimes}D^1(U) \quad , \quad \beta\in\overset{s}{\otimes}D_1(U) \quad , \quad \gamma\in\overset{r}{\otimes}D^1(U)\times\overset{s}{\otimes}D_1(U)$$

are called **covariant isotensor field of order r, contravariant isotensor field of order s and isotensor field of type (r,s), respectively.**

Let $(x_1,\dots,x_n)$ be the local isocoordinates on U. It is known that the isovector fields :

$$\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}$$

form a base of the isomodule $D^1(U)$ and the isoexterior 1-form :

$$dx_1,\dots,dx_n$$

form a base of the isomodule $D_1(U)$. It can be easily obtained that :

$$\left\{ \frac{\partial}{\partial x_{i_1}} \otimes \frac{\partial}{\partial x_{i_2}} \otimes \ldots \otimes \frac{\partial}{\partial x_{i_r}} \otimes dx_{j_1} \otimes dx_{j_1} \otimes \ldots \otimes dx_{j_r} \right\}$$

where $1 \le i_1, i_2, \ldots, i_r \le n$ and $1 \le j_1, j_2, \ldots, j_s \le n$ form a base of the isomodule :

$$\hat{\otimes} D^1(U) \times \hat{\otimes} D_1(U)$$

whose dimension is n^{r+s}.

4.6.5. Differentiability of the isotensor fields

If we use $D^0(M), \mathcal{C}^k$, $D^1(M), \mathcal{C}^k$ and $D_1(M), \mathcal{C}^k$ instead of $D^0(M)$, $D^1(M)$ and $D_1(M)$ respectively, then we can obtain covariant isotensor fields, contravariant isotensor fields and mixed isotensor field of class $\mathcal{C}^k$ denoted by :

$\hat{\otimes} D_1(M), \mathcal{C}^k$ covariant isotensor field of order r of class $\mathcal{C}^k$

$\hat{\otimes} D^1(M), \mathcal{C}^k$ contravariant isotensor field of order s of class $\mathcal{C}^k$

$\hat{\otimes} D_1(M) \hat{\otimes} D^1(M), \mathcal{C}^k$ mixed isotensor field of type (r,s) of class $\mathcal{C}^k$. Usually we work on isotensor field of class $\mathcal{C}^\infty$.

NEW NOTATION 4.6.6 The isomodule $\overset{r}{\otimes}D_1(M)\overset{s}{\otimes}D^1(M)$ over the isoalgebra $D^0(M)$ is denoted by $D^r_s(M)$, that is,

$$\overset{r}{\otimes}D_1(M)\overset{s}{\otimes}D^1(M) = D^r_s(M)$$

and the same time we use the symbols :

$$\overset{r}{\otimes}D^1(M) = D^r_0(M) = D^r(M)$$

$$\overset{s}{\otimes}D_1(M) = D^0_s(M) = D_s(M)$$

UNIFICATION 4.6.7 Let M be a diffetentiable isomanifold of dimension n. From M we obtain the sets :

$$D^0(M)\ ,\ D^1(M)\ ,\ D_1(M)\ ,\ D^r(M)\ ,\ D_s(M)\ \text{ and }\ D^r_s(M)$$

of the isofunction, isovector fields, isoexterior 1-forms and isotensor fields of type (r,s) respectively. Now we can consider each isofunction as an isotensor fields of type (0,0). We also consider an isovector field and an isoexterior 1-form as isotensor fields of type (1,0) and (0,1) respectively.

Each contravariant isotensor field of order r can be identified as isotensor or field of type (r,0). Also each covariant isotensor field of order s can be identified as isotensor field of type (0,s)

4.6.8. Another definition of the isotensor field Let M be a differentiable isomanifold of dimension n. Let $T_p(M)$ be the isotangent space of M at the point P to whom we associate the isocontangent space $T^*_p(M)$, which are isovector space of

dimension n, that is,

$$\dim(T_p(M)) = \dim(T_p^*(M)) = n$$

It is known that from the isovector space $T_p(M)$ we obtain the isotensor algebra :

$$T(T_p(M)) = {}^{0}\hat{\otimes}^{\infty}_{r,s} \left(\hat{\otimes} T_p(M) \hat{\otimes} T_p^*(M) \right)$$

under the condition :

$$\hat{\otimes} T_p(M) \hat{\otimes} T_p^*(M) = \hat{\mathbb{R}}$$

If we consider the isovector space $\hat{\otimes} T_p(M) \hat{\otimes} T_p^*(M)$ then each element of this isovector space is called isotensor of type (r,s) obtained from the isovector space $T_p(M)$.

Let A be an isotensor field of type (r,s) on the differentiable isomanifold. This isotensor field A is characterized by n^{s+r} isofunction on M. Therefore, A(P) consists of a set of n^{r+s} numbers. In this case A(P)=o for some o of the isovector

$$\hat{\otimes} T_p(M) \hat{\otimes} T_p^*(M)$$

Hence A(P) is an isotensor of type (r,s) obtained by the isovector space $T_p(M)$.

From the above we conclude that every isotensor field of type (r,s) can be considered as a set of isotensors of type (r,s). To each point P of the differentiable isomanifold M corresponts one isotensor corresponding to

$T_p(M)$.

PROPOSITION 4.6.4 Let $D^r_s(M)$ and $D^s_r(M)$ be the two isomodules of the isotensor fields of type (r,s) and (s,r) respectively, where M is differentiable isomanifold.

Proof The isomanifold $D^r_s(M)$ is defined by :

$$D^r_s(M) = \left\{ f \;/\; f : \underbrace{D_1 \times \ldots \times D_1}_{r\text{-times}} \times \underbrace{D^1 \times \ldots \times D^1}_{s\text{-times}} \longrightarrow D^0(M) \;, \right.$$

$$\left. f \text{ (r+s)-multilinear form} \right\} \qquad (4.6.1)$$

The isomodule $D^s_r(M)$ is defined dy :

$$D^s_r(M) = \left\{ \varphi \;/\; \varphi : \underbrace{D_1 \times \ldots \times D_1}_{s\text{-times}} \times \underbrace{D^1 \times \ldots \times D^1}_{r\text{-times}} \longrightarrow D^0(M) \;, \right.$$

$$\left. \varphi \text{(s+r) multilinear form} \right\} \qquad (4.6.2)$$

From the relations (4.6.1) and (4.6.2) we conclude that each element of $D^r_s(M)$ can be considered as a linear form on D^s_r, that means the $D^r_s(M)$ is a dual space of D^s_r, that means :

$$D^r_s(M) = (D^s_r(M))^*$$

DEFINITION 4.6.5. of the isotensor algebra associated to an isomanifold. Let M be a differentiable isomanifold. From this isomanifold we obtain the isoalgebra $D^0(M)$ and the

isomodules :

$$D_s^r(M) \quad r,s=0,1,\ldots$$

with the constrains :

$$D_0^0(M)=D^0(M) \ , \ D_0^r(M)=D^r(M) \ , \ D_s^0(M)=D_s(M) \qquad (4.6.3)$$

We consider the direct sum :

$$D(M) = D^0(M)\oplus D^1(M)\oplus D_1(M)\oplus D_1^1(M)\oplus D_2^0(M)\oplus\ldots$$

which by means of (5.3) can be written :

$$D(M) = {}^0\!\oplus_{r,s}^{\infty} D_s^r(M)$$

$D(M)$, as a direct sum of isomodules over $D^0(M)$, becomes an isomodule over the isoalgebra $D^0(M)$.

On this isomodule $D(M)$ we define a new internal law of composition denoted by $\circledast$ and called isotensor product. If S and T are two isotensor fields of type (λ,μ) and (ν,ρ) respectively, their isotensor product is defined as follows

$$(S\circledast T)_P = S_P\circledast' T_P \qquad \forall P\in M \qquad (4.6.4)$$

where $\circledast'$ is the isotensor product on the isotensor algebra :

$$^n\!\oplus_{r,s}^{\infty} \hat{\otimes} T_P(M)\hat{\otimes} T_P^*(M) \qquad (4.6.5)$$

We must remark that the relation (4.6.4) define completely the$(S\circledast T)_P$ because its second member is the isotensor of S_P and T_P. Since the isotensor fields S and T are known, hence S_P and T_P are also known for every point $P\in M$. Therfor, the

isotensor product $S_P \circledast \cdot T_P$ is known. The relation (4.6.4) determine the values of $S \circledast T$ for every point P of M and hence $S \circledast T$ is known.

The isotensor product $\circledast \cdot$ on the isomodule D(M) turns of it an isoalgebra which is called **isotensor algebra associated to the isomanifold.**

4.6.9 Special isotensor field Let M be a differentiable isomanifold from which we obtain the isotensor algebra D(M). There are special isotensor fields. We mention some of them.

4.6.10 Kronecker's isotensor field The isotensor field δ of type (1.1) on the isomanifold M is characterized with the property : If $(\delta_1^1, \ldots, \delta_n^1, \delta_1^2, \ldots, \delta_n^n)$ are its isocoordinates then we have :

$$\delta_i^j(P) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases} \qquad \forall P \in M$$

If we take the isotensor product of δ by itself k times, then we obtain the isotensor field :

$$\underbrace{\delta \circledast \delta \circledast \ldots \circledast \delta}_{k\text{-times}} = \overset{k}{\circledast} \delta$$

wich has isocoordinates $\left(\delta_{j_1 \ldots j_k}^{i_1 \ldots i_k}\right)$ defined by :

$$(\hat{\delta})_p = \left(\hat{\delta}^{i_1 \ldots i_k}_{j_1 \ldots j_k}\right)_p = \begin{cases} 1 & \text{if } i_1, \ldots, i_k \text{ is even permutation of } j_1, \ldots, j_k \\ -1 & \text{if } i_1, \ldots, i_k \text{ is odd permutation of } j_1, \ldots, j_k \\ 0 & \text{in all other cases} \end{cases}$$

4.6.11 Symmetric covariant isotensor field. Let K be an isotensor field of type (0,s) on the isomanifold M, that means K is a covariant isotensor field of order s on M and that implies :

$$K \in D_s(M)$$

Therefore, K can be considered as an s multilinear form on $D^1(M)$, that is :

$$K : D^1 \times D^1 \times \ldots \times D^1 \longrightarrow D^0$$
$$K : (X_1, X_2, \ldots, X_s) \longrightarrow \phi(X_1, X_2, \ldots, X_s)$$

If we have :

$$K(X_1, X_2, \ldots, X_i, \ldots, X_j, \ldots, X_s) = K(X_1, X_2, \ldots, X_j, \ldots, X_i, \ldots, X_s)$$

then the s-multilinear isoform is called **symmetric with respect to the indices** i and j. If K is symmetric with respect to all indices then it is called **symmetric s-multilinear isoform** or **symmetric covariant isotensor field of order s.**

4.6.12 Symmetric contravariant isotensor field. Let Π be an isotensor field of type $(r,0)$ on the isomanifold M, that is

$$\Pi \in D^r(M)$$

Π is an r-multilinear isoform on D_1 :

$$\Pi : \underbrace{D_1 \times \ldots \times D_1}_{r\text{-times}} \longrightarrow D_0$$

$$\Pi : (w_1, \ldots, w_r) \longrightarrow \Pi(w_1, \ldots, w_r)$$

If we have :

$$\Pi(w_1, \ldots, w_i, \ldots, w_j, \ldots, w_r) = \Pi(w_1, \ldots, w_j, \ldots, w_i, \ldots, w_r)$$

then Π is called symmetric **s-multilinear isoform** on D_1 with respect to the indices i and j. If Π is symmetric with respect to all indices, then it is called **symmetric contravariant isotensor field of order s.**

REMARK 4.6.1 If K is a symmetric covariant field of order s, then we have :

$$K(x_1, \ldots, x_s) = K(x_{\sigma(1)}, \ldots, x_{\sigma(s)})$$

where $\sigma \in J_s(1, \ldots, s)$ and $J_s 1, \ldots, s)$ is the set of permutations of $1, \ldots, s$.

If Π is a symmetric contravariant isitensor field of order r, then we obtain :

$$\Pi(w_1, \ldots, w_r) = \Pi(w_{\sigma(1)}, \ldots, w_{\sigma(r)})$$

where $\sigma \in J_r(1, \ldots, r)$.

4.6.13 Antisymmetric covariant isotensor field. Let A be a covariant isotensor field of order r, that means :

$$A : D^1 \times \ldots \times D^1 \longrightarrow D_0 ,$$

$$A : (x_1, \ldots, x_r) \longrightarrow A(x_1, \ldots, x_r)$$

in other words A is an r-multilinear isoform on D^1. If we have

$$A(x_1, x_2, \ldots, x_i, \ldots, x_j, \ldots, x_r) =$$
$$-A(x_1, x_2, \ldots, x_j, \ldots, x_i, \ldots, x_r)$$

then A is called **antisymmetric r-multilinear isoform with respect to the indices i and j**. If A is a antisymmetric with respect to any succesive pair of indices, then A is called **antisymmetric covariant isotensor field of order r**. For the A, we have :

$$A(x_1, \ldots, x_r) = T(o)A(x_{\sigma(1)}, \ldots, x_{\sigma(r)})$$

where $\sigma \in J_r(1, \ldots, r)$ and

$$T(o) = \begin{cases} 1 \text{ if } \sigma \text{ is even permutation} \\ -1 \text{ if } \sigma \text{ is odd permutation} \end{cases}$$

REMARK 4.6.2 If A is an antisymmetric covariant isotensor field of order r, then

$$A(x_1, \ldots, x_r) = 0$$

if two of $x_1,\ldots,x_r$ are propotional.

REMARK 4.6.3 With the same manner we can define an anti-symmetric contravariant isotensor field.

REMARK 4.6.4 Let K be a symmetric covariant isotensor field of order s on the isomanifold M. For each P∈M, we have a symmetric covariant isotensor K_p obtain by the isotangent space $T_p(M)$

4.7. SUBALGEBRAS OF THE ISOTENSOR ALGEBRA ASSOCIATED TO AN ISOMANIFOLD

4.7.1 Subalgebra of the covariant isotensor fields. Let M be a differentiable isomanifold. From this we obtain the **a**iso-modules $D_1(M)$, $D_2(M),\ldots D_r(M),\ldots$ over $D^0(M)$. We consider the direct sum :

$$sD(M) = D_0(M)\oplus D_1(M)\oplus\ldots = \bigoplus_{r=0}^{\infty} D_r(M)$$

under the notation $D_0(M)\neq D^0(M)$. It can be easily proved that sD(M) is an subalgebra of the isotensor algebra :

$$D(M) = \bigoplus_{r,s=0}^{\infty} D_s^r(M)$$

Since D(M) is an isotensor algebra we obtain that sD(M) is an isotensor subalgebra of D(M), which is called **covariant**

isotensor algebra associated to **M**.

4.7.2 Subalgebra of the contravariant isotensor fields. In the same manner as in 4.7.1 we obtain the subalgebra :

$$\alpha D(M) = D^0(M) \oplus D^1(M) \oplus \ldots = \bigoplus_{r=0}^{\infty} D^r(M)$$

of the D(M), which is called **contravariant isotensor algebra associated to M**.

4.8. MAPPINGS ON THE ISOTENSOR ALGEBRA D(M)

4.8.1 Symmetrization mapping. Let sD(M) be the covariant isotensor algebra associated to the isomanifold M. We consider the mapping :

$$S : sD(M) \longrightarrow sD(M)$$

$$S : K \longrightarrow S(K)$$

which is defined as follows : If K is a covariant isotensor field of orde r, then we assume that S(K) is also a covariant isotensor field of order r. K is an r-multilinear isoform :

$$K : D^1 \times \ldots \times D^1 \longrightarrow D^0$$

$$K : (x_1, \ldots, x_r) \longrightarrow K(x_1, \ldots x_r)$$

and the same for S(K), that is :

$$S(K) : D^1 \times \ldots \times D^1 \longrightarrow D^0$$

$$S(K) : (x_1,\ldots,x_r) \longrightarrow S(K)(x_1,\ldots x_r)$$

where $S(K)(x_1,\ldots,x_r)$ is defined by:

$$S(K)(x_1,\ldots,x_r) = \frac{1}{r!} \sum_{\sigma\in\Delta_r(J_r)} K(x_{\sigma(1)},\ldots,x_{\sigma(r)})$$

where σ is one of the permutations of $[1,\ldots,r]=J_r$ and $\Delta_r(J_r)$ is the set of all permutations obtained from J_r.

It can be easily proved that the mapping S is a linear on the sD(M) and is called **symmetrization on the covariant isotensor algebra sD(M) associated to M.**

REMARK 4.8.1 On the contravariant isotensor algebra oD(M) associated to an isomanifold M we can define in a similar manner as in 4.8.1 the symmetrization mapping.

4.8.2 Alternation mapping. Let sD(M) be the covariant isotensor algebra associated to the isomanifold M. We define the mapping :

$$A : sD(M) \longrightarrow sD(M) \qquad A : N \longrightarrow A(N)$$

as follows : If N is a contravariant isotensor field of order r, then A(N) is also a contravariant isotensor field of order r. N can be considered as r-multilinear isoform on $D^1(M)$ and therefore, ew have :

$$N : D^1 \times\ldots\times D^1 \longrightarrow D^0$$

$$N : (x_1,\dots,x_r) \longrightarrow N(x_1,\dots,x_r)$$

Similarly for the isotensor field A(N) we have :

$$A(N) : D^1 x \dots x D^1 \longrightarrow D^0$$

$$A(N) : (x_1,\dots,x_r) \longrightarrow A(N)(x_1,\dots x_r)$$

We difine $A(N)(x_1,\dots,x_r)$ as follows :

$$A(N)(x_1,\dots,x_r) = \frac{1}{r!} \sum_{\sigma\in\Delta(J_r)} \varepsilon(o)\ N(x_{\sigma(1)},\dots,x_{\sigma(r)})$$

where o is a permutation of the set $J_r=[1,\dots,r]$ and $\Delta_r(J_r)$ is the set of permutations of J_r and

$$\varepsilon(o) = \begin{cases} 1 \text{ if o is an even permutation} \\ -1 \text{ if } \sigma \text{ is an odd permutation} \end{cases}$$

The mapping A, which is linear, is called **altirnation.**

REMARK 4.8.2 In the same manner we define the altirnation mapping on the contravariant isotensor algebra oD(M).

4.8.3 Meaning of the linear maps S and A. If K is a covariant isotensor field of order r, then S(K) is a symmetric covariant isotensor field of order r. The symmetry is obtained by the definition of the symmetrization mapping. Therefore, S is an operation which sends each covariant isotensor filed to another symmetric covariant isotensor field.

Let N be a covariant isotensor field of order r. A(N) is an

antisymmetric covariant isotensor field of order r. Hence the alternation mapping A is an operation which sends each covariant isotensor field to another antisymmetric covariant isotensor field.

4.8.4 Contraction. Let D(M) be the isotensor algebra associated to the isomanifold M. Let $D^r_s(M)$ and $D^{r-1}_{s-1}(M)$ be the isomodules over the isoalgebra $D^0(M)$. We consider the mapping C^i_j between them defined by :

$$C^i_j : D^s_r(M) \longrightarrow D^{s-1}_{r-1}(M) \quad , \quad C^i_j : N \longrightarrow C^i_j(N)$$

The mapping is completely determined if we know $C^i_j(N)$. It is known that N is (r+s)-multilinear isoform, that is :

$$N : \underbrace{D^1 x \ldots x D^1}_{r\text{-times}} x \underbrace{D_1 x \ldots x D_1}_{s\text{-times}} \longrightarrow D^0$$

$$N : (x_1,\ldots,x_r,w_1,\ldots,w_s) \longrightarrow N(x_1,\ldots,x_r,w_1,\ldots,w_s)$$

The mapping C^i_j is completely defined as follows :

$$C^i_j : x_1 \otimes \ldots \otimes x_s \otimes w_1 \otimes \ldots \otimes w_r \longrightarrow C^i_j(x_1 \otimes \ldots \otimes x_s \otimes w_1 \otimes \ldots \otimes w_r) =$$

$$= w_j(x_i)(x_1 \otimes \ldots \otimes x_{i-1} \otimes x_{i+1} \otimes \ldots \otimes x_s \otimes w_1 \otimes \ldots \otimes w_{j-1} \otimes w_{j+1} \otimes \ldots \otimes w_r)$$

From the definition of the contraction mapping C^i_j we can easily prove that this mapping is linear between the two

isomodule $D_r^s(M)$ and $D_{r-1}^{s-1}(M)$. Given the isomodule $D_r^s(M)$ of all isotensor fields of type (s,r) we can determine for every pair (i,j) $i\leq s$ and $r\leq j$ the linear mapping C_j^i which is called **contraction** between the isomodules $D_r^s(M)$ and $D_{r-1}^{s-1}(M)$.

CHAPTER V

ISOEXTERIOR ALGEBRA

5.1 CONSTRUCTION OF THE ISOEXTERIOR ALGEBRA ASSOCIATED TO AN ISOMANIFOLD

In the preceding two chapters we have studied the isotopies of manifolds. In this chapter we shall then study the isotopies of the exterior calculus on manifolds. This study was initiated by Santilli (1991b) who : 1) achieved the rudiments of the exterior calculus on isofields, called **isoexterior calculus**, which permits a nontrivial integral generalization of the conventional one, under the condition that all integral terms are embedded in the isounit I^* of the base isofield $\mathbb{F}^*$; 2) identified the isotopies of exact p-forms; 3) proved the existence of the isotopic Poincare' Lemma, i.e. the preservation under isotopies of the original geometric axioms (exact p-forms are closed, whether conventional or isotopic); and then applied these broader methods to the tretment of nonlinear, nonlocal and nonhamiltonian systems.

In this chapter we shall use the above results and incorporate them in more formal unpublished treatments.

DEFINITION of isoexterior product 5.1.1. Let **M** be a differentiable isomanifold of dimension n. We consider the

isomodule $D_r(M)$ over the isoalgebra $D^0(M)$, which consists of contravariant isotensor fields of order r. From $D_r(M)$ we obtain its subset $\Lambda^r(M)$ of antisymmetric contravariant isotensor field of order r, that is :

$$\Lambda^r(M) = \left\{ w \in D^r(M) \;/\; A(w)=w \right\}$$

PROPOSITION 5.1.1 The subset $\Lambda^r(M)$ of $D^r(M)$ is a subisomanifold.

Poof : If $w_1, w_2 \in \Lambda^r(M)$, then we have :

$$A(w_1)=w_1 \qquad A(w_2)=w_2 \tag{5.1.1}$$

Since A is a linear mapping, then we obtain :

$$A(w_1+w_2) = A(w_1)+A(w_2) \tag{5.1.2}$$

which, by means of (5.1.1), takes the form :

$$A(w_1+w_2) = w_1+w_2$$

Hence Λ^r is a subisomodule of $D_r(M)$.

REMARK 5.1.1 From the construction of $\Lambda^r(M)$ we obtain that :

$$\Lambda^r(M) = 0 \quad \text{if } r>n \quad \text{where } n=\dim M$$

5.1.2 <u>Construction of the isomodule $\Lambda^r(M)$.</u> Let **M** be a differentiable isomanifold. From this we obtain the isomodules :

$$\Lambda^0(M)=D^0(M) \quad , \quad \Lambda^1(M) \quad , \quad \Lambda^2(M) \quad , \ldots , \quad \Lambda^r(M)$$

over the isoalgebra $D^0(M)=\Lambda^0(M)$. We consider the direct sum of these isomodules :

$$\Lambda(M) = \Lambda^0(M)\oplus\Lambda^1(M)\oplus\Lambda^2(M)\oplus\ldots\oplus\Lambda^n(M) = \bigoplus_{r=0}^{n} \Lambda^r(M)$$

THEOREM 1.5.1 The isomodule $\Lambda(M)$ can become an isoalgebra.

Proof : On this isomodule $\Lambda(M)$ we define a new law of internal composition denoted by $\wedge$ as follows :

$$\wedge : \Lambda(M)\times\Lambda(M) \longrightarrow \Lambda(M)$$

$$\wedge : (w_1,w_2) \longmapsto w_1\wedge w_2$$

where $w_1\wedge w_2$ is defined by :

$$w_1\wedge w_2 \equiv A(w_1\otimes w_2)$$

that means $w_1\wedge w_2$ is equal to the alternation of the isotensor product of the two isotensor fields w_1 and w_2. This law has the associative property, that means :

$$w_1\wedge(w_2\wedge w_3) = (w_1\wedge w_2)\wedge w_3 \quad \forall\ w_1,w_2,w_3\in\Lambda(M) \qquad (5.1.3)$$

The isomodule $\Lambda(M)$ over $D^0(M)$ with the internal composition $\wedge$ becomes an algebra over the isoalgebra $\Lambda^0(M)$.

DEFINITION of the isoexterior isoalgebra 5.1.2 Le **M** be a differentiable isomanifold. The isoalgebra $\Lambda(M)$ is called **isoexterior algebra associated to M.**

PROPOSITION 5.1.2 Let M be a differantiable isomanifold. Then $\Lambda(M)$ is a subalgebra of the isotensor algebra $D(M)$ associated to M.

Proof: The elements of $\Lambda(M)$ are antisymmetric covariant isotensor fields and therefore belong to $D(M)$. Hence

$$\Lambda(M) \subset D(M)$$

and since $w_1 \wedge w_2 = A(w_1 \otimes w_2)$ we obtain $\Lambda(M)$ is a subalgebra of $D(M)$.

DEFINITION of isoexterior form 5.1.3 The isoexterior algebra $\Lambda(M)$ of a differentiable isomanifold M can be written :

$$\Lambda(M) = \Lambda^0(M) \oplus \Lambda^1(M) \oplus \ldots \oplus \Lambda^n(M) = \bigoplus_{r=1}^{n} \Lambda^r(M)$$

The elements of $\Lambda^r(M)$ $r=0,1,\ldots,n$ is called **isoexterior form of order r** or **isoexterior r-form on M.**

REMARK 5.1.1 From the construction of the isoexterior algebra we obtain that the isofunctions on M are isoexterior o-forms on M.

PROPOSITION 5.1.3 If w_1 and w_2 are two isoexterior forms of order r and s respectively, then $w_1 \wedge w_2$ is an isoexterior of order r+s. The following relation :

$$w_1 \wedge w_2 = (-1)^{rs} w_2 \wedge w_1$$

is hold.

Proof : If we take $w_1 \in \Lambda^r(M)$ and $w_2 \in \Lambda^s(M)$ then we have :

$$w_1 \wedge w_2 = A(w_1 \otimes w_2)$$

Since w_1 and w_2 are antisymmetric contravariant isotensor fields of order r and s respectively, then

$$w_1 \otimes w_2 \in D_{r+s}(M)$$

is an antisymmetric covariant isotensor field of order r+s. Therefore, we have :

$$w_1 + w_2 \in \Lambda^{r+s}(M)$$

From this we get that the isoexterior product $\wedge$ is a mapping

$$\wedge : \Lambda^r(M) \times \Lambda^s(M) \longrightarrow \Lambda^{r+s}(M)$$

PROPOSITION 5.1.4 If $w_1 \in \Lambda^r(M)$ and $w_2 \in \Lambda^s(M)$, then we obtain :

$$w_1 \wedge w_2 = (-1)^{rs}\, w_2 \wedge w_1$$

Proof : It is known that :

$$w_1 \wedge w_2 \in \Lambda^{r+s}(M) \quad , \quad w_2 \wedge w_1 \in \Lambda^{r+s}(M)$$

and

$$w_1 \wedge w_2 = A(w_1 \otimes w_2) \quad , \quad w_2 \wedge w_1 = A(w_2 \otimes w_1)$$

From the definition of isoexterior product we obtain :

$$w_1 \wedge w_2(X_1, \ldots, X_{r+s}) = \frac{1}{(r+s)!} \sum_{\sigma \in J_{r+s}} J(\sigma) w_1(X_{\sigma(1)}, \ldots, X_{\sigma(r)}) \cdot$$
$$\cdot\, w_2(X_{\sigma(r+1)}, \ldots, X_{\sigma(r+s)}) \tag{5.1.4}$$

$$w_2 \wedge w_1(x_1,\ldots,x_{r+s}) = \frac{1}{(r+s)!} \sum_{\sigma \in J_{r+s}} w_2(x_{\sigma(1)},\ldots,x_{\sigma(s)}) \cdot$$

$$\cdot\, w_1(x_{\sigma(s+1)},\ldots,x_{\sigma(s+t)}) \qquad (5.1.5)$$

The relations (5.1.4) and (5.1.5) imply that the (5.1.5) is obtained from (5.1.4) if we make rs permutations of $\{1,\ldots,r,r+1,\ldots,r+s\}$. This implies the relation :

$$w_1 \wedge w_2 = (-1)^{rs} w_2 \wedge w_1$$

THEOREM 5.1.1 Let M be a differentiable isomanifold of dimension n. The isomodule $\Lambda^r(M)$ $1 \le r \le n$ over the isoalgebra $\Lambda^0(M)$ is of dimension $\binom{n}{r}$.

Proof : It is known that the isomodule $\Lambda^1(M)$ has dimension n. If $w_1,\ldots,w_n$ is a base of $\Lambda^1(M)$, then the isoexterior 2-forms :

$$w_1 \wedge w_2,\ldots,w_1 \wedge w_n, w_2 \wedge w_3,\ldots,w_2 \wedge w_n,\ldots,w_{n-1} \wedge w_n$$

form a base for the isomodule $\Lambda^2(M)$. Therefore, the dimension of $\Lambda^2(M)$ is $\binom{n}{2}$. With the same method we can prove that the isoexterior r-forms :

$$w_{i_1} \wedge w_{i_2} \wedge \ldots \wedge w_{i_r} \qquad (5.1.6)$$

where $1 \le i_1 < i_2 < \ldots < i_r \le n$ is a base of $\Lambda^r(M)$. The number of these isoexterior r-forms is $\binom{n}{r}$.

Another DEFINITION of the isoexterior r-form (5.1.4). Let $T_p(M)$ be the isotangentspace of the differentiable

isomanifold M at the point P. From this isovector space $T_p(M)$ we obtain the isoexterior algebra :

$$\Lambda(T_p(M)) = \overset{n}{\underset{r=1}{\oplus}} \Lambda^r(T_p(M))$$

where r=dimM. The internal law of composition on this isovector space $\Lambda(T_p(M))$, which makes it an isoalgebra, is the isoexterior product.

The isoexterior r-form w on the differentiable isomanifold M is define as follows :

$$w_p \in \Lambda^r(T_p(M))$$

From this definition we have the formula :

$$(w_1)_p \wedge (w_2)_p = (w_1 \wedge w_2)_p \qquad (5.1.7)$$

where $\wedge$ in the first member of (5.1.7) means the isoexterior product on $\Lambda(T_p(M))$ and the $\wedge$ the isoexterior product on $\Lambda(M)$.

5.2 LOCAL EXPRESSION OF AN ISOEXTERIOR R-FORM

5.2.1 **Isoexterior algebra on an isochart** : Let (U,ϕ) be an isochart of a differentiable isomanifold with a local isocoordinate system $(x^1,\ldots,x^n)$. Previously we have defined the isoexterior algebra on the differentiable isomanifold M. In a semilar manner we can define the isoexterior isoalgebra $\Lambda(U)$ on U, which is a subalgebra of the isotensor algebra D(U). This isotensor algebra $\Lambda(U)$ can be defined as follows

$$\Lambda(U) = \Lambda^0(U)\oplus\Lambda^1(U)\oplus\ldots\oplus\Lambda^n(U) \qquad (5.2.1)$$

where $\Lambda^0(U)=D^0(M)$ is the isoalgebra of the isofunctions on U, and n the dimension of M, $\Lambda^1(U)$ the isomodule of isoexterior r-form on U and $\Lambda^k(U)$ k=1,...,n the isomodule of isoexterior k-form on M, k=1,...,n.

5.2.2 Isoexterior 1-form on the isochart. Every isoexterior 1-form w on the isochart (U,φ) with local isocoordinate system $(x^1,\ldots,x^n)$ takes the form :

$$w = f_1{*}dx^1+f_2{*}dx^2+\ldots+f_n{*}dx^n = \underset{\lambda=1}{\overset{n}{\oplus}} w_\lambda{*}dx^\lambda$$

where $f_i{*}dx^i = f_i T dx^i$ with $T^{-1} = I^* =$ isounit of $\mathbb{F}^*$ and $f_1,f_2,\ldots,f_n \in D^0(U)$. The isomodule $\Lambda^1(U)$ is of dimension n over $D^0(M)$. The dx^1, dx^2, ... , dx^n are isoexterior 1-forms forming one base of $\Lambda^1(M)$. This base is the isodual of the base :

$$\frac{\partial}{\partial x^1}, \; \frac{\partial}{\partial x^2}, \; \ldots, \; \frac{\partial}{\partial x^n}$$

of the isomodule $D^1(M)$ of all isovector fields on U. Therefore, we have :

$$dx^l\left(\frac{\partial}{\partial x^k}\right) = \delta^l_k = \begin{cases} 1 & \text{if } l=k \\ 0 & \text{if } l\neq k \end{cases}$$

If $w_1,w_2,\ldots,w_n$ are isoexterior 1-forms on U, then these can be written :

$$w_i = \sum_{j=1}^{n} f_{ij}dx^j$$

The isoexterior 1-forms $w_1,w_2,\ldots,w_n$ form a base of $\Lambda^1(U)$ is the matrix :

$$\Lambda = \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ f_{n1} & f_{n2} & & f_{nn} \end{pmatrix}$$

is order n or equivalently the determinant :

$$\begin{vmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ f_{n1} & f_{n2} & & f_{nn} \end{vmatrix} = D(A) \neq 0$$

where f_{ij} $1 \leq i,j \leq n$ are differentiable isofunction on U.

5.2.3 Isoexterior 2-form on an isochart. We consider an isoexterior 2-form w on U. Therefore, w can be written :

$$w = w_{12}dx^1\wedge dx^2+\ldots+w_{1n}dx^1\wedge dx^n+w_{2n}dx^2\wedge dx^n+\ldots+w_{2n}dx^2\wedge dx^n+$$

$$+ \ldots + w_{n-1\ n}dx^{n-1}\wedge dx^n = \sum_{k<l}^{1,n} w_{kl}dx^k\wedge dx^l$$

where $w_{kl}\in D^0(U)$

5.2.4 Isoexterior k-form on an isochart k≥3. The isoexterior k-form w on the isochart (U,ϕ) takes the form :

$$w = \sum f_{i_1\ldots i_k} dx^{i_1}\wedge\ldots\wedge dx^{i_k}$$

$1\le i_1<i_2<\ldots<i_k\le n$ and $f_{i_1\ldots i_k}$ are the isofunctions on U. It is obvious that $w\in\Lambda^k(U)$

The isoexterior k-forms :

$$dx^{i_1}\wedge\ldots\wedge dx^{i_k} \quad 1\le i_1<i_2<\ldots<i_k\le n$$

form a base of the isomodule $\Lambda^k(U)$ and hence the dimension of $\Lambda^k(U)$ is $\binom{n}{k}$.

5.3 OPERATOR ON AN ISOALGEBRA

5.3.1 Isoexterior differentiation Let $\Lambda(M)$ be the isoexterior algebra on the differentiable isomanifold. It is known that $\Lambda(M)$ can be written :

$$\Lambda(M) = \Lambda^0(M)\oplus\Lambda^1(M)\oplus\ldots\oplus\Lambda^n(M)$$

We define a mapping d :

$$d : \Lambda^r(M) \longrightarrow \Lambda^{r+1}(M) \qquad d : w \longmapsto dw$$

for which we assume that it is linear :

$$d : \lambda_1 w_1+\lambda_2 w_2 \longmapsto d(\lambda_1 w_1+\lambda_2 w_2) = \lambda_1 dw_1+\lambda_2 dw_2$$

where $\lambda_1,\lambda_2\in\hat{\mathbb{R}}$ the isofield and $w_1,w_2\in\Lambda^r(M)$ and has also the properties :

(I) $$d\circ d=0 \tag{5.3.1}$$

(II) $$d(w_1\wedge w_2) = dw_1\wedge w_2+(-1)^r w_1 dw_2 \tag{5.3.2}$$

The linear mapping d with these properties is called **isoexterior differentiation.**

PROPOSITION 5.3.2. The isoexterior differentiation d on the isoexterior algebra $\Lambda(M)$ is a differentiation of degree 1.

Proof : From the construction of d we conclude that d has the property :

$$d : \Lambda^r(M) \longrightarrow \Lambda^{r+1}(M)$$

and since it satisfies the relation (5.3.2) we conclude that d is a differentiation of degree one.

5.3.3 The isoexterior differentiation on an isochart. Let (U,φ) be a chart with local isocoordinate $(x^1,\dots x^n)$. If f is an isofunction on U, then we have :

$$df = \frac{\partial f}{\partial x^1}dx^1 + \frac{\partial f}{\partial x^2}dx^2 + \dots + \frac{\partial f}{\partial x^n}dx^n$$

If w is an isoexterior 1-form on U, then this can be written

$$w = f_1 dx^1 + f_2 dx^2 + \dots + f_n dx^n$$

where $f_1, f_2, \dots, f_n$ isofunctions on U. Then we obtain :

$$dw = \left(\frac{\partial f_2}{\partial x^1} - \frac{\partial f_1}{\partial x^2}\right)dx^1 \wedge dx^2 + \left(\frac{\partial f_3}{\partial x^1} - \frac{\partial f_1}{\partial x^3}\right)dx^1 \wedge dx^3 + \dots +$$

$$+ \left(\frac{\partial f_n}{\partial x^1} - \frac{\partial f_1}{\partial x^n}\right)dx^1 \wedge dx^n + \dots + \left(\frac{\partial f_n}{\partial x^{n-1}} - \frac{\partial f_n}{\partial x^n}\right)dx^{n-1} \wedge dx^n$$

Finally if w is an isoexterior r-form, then w has the form on U :

$$w = \sum_{1 \le i_1 < i_2 < \dots < i_r < n} w_{i_1 \dots i_r} dx^{i_1} \wedge \dots \wedge dx^{i_r} \qquad (5.3.3)$$

where $w_{i_1 \dots i_r}$ are isofunctions on U. The relation (5.3.3) implies :

$$w = \sum_{\substack{i \notin \{i_1, \dots, i_r\} \\ 1 \le i_1 < i_2 < \dots < i_r}} \frac{\partial w_{i_1 \dots i_r}}{\partial x^i} \partial x^{i_1} \wedge \dots \wedge dx^{i_r} \wedge dx^i$$

$i \in \{1,\ldots,n\}$

THEOREM 5.3.1 If w is an isoexterior r-form on a defferentiable isomanifold, then we have :

$$dw(x_0,x_1,\ldots,x_r) = \frac{1}{r+1} \sum_{l=0}^{r} (-1)^l x_i\left(w(x_0,\ldots,\hat{x}_l,\ldots,x_r)\right) +$$

$$+ \frac{1}{r+1} \sum_{0 \le l < k \le r} (-1)^{l+k} w\left([x_l,x_k],x_0,\ldots,\hat{x}_l,\ldots,\hat{x}_k\right)$$

where the symbol ^ means the term is ommited. For an isoexterior 1-form w and an isoexterior 2-form φ we have :

$$(dw)(X,Y) = \frac{1}{2}\left[Xw(Y)-Y(w(X))-w([X,Y]) \right]$$

$$(d\varphi)(X,Y,Z) = \frac{1}{3}\left[X(\varphi(Y,Z))+Y(\varphi(Z,X))+Z(w(X,Y))\right] -$$

$$- \varphi([X,Y],Z) - \varphi([Y,Z],X) - \varphi([Z,X],Y)$$

where X,Y,Z are isovector fields on M

Proof : The proof is by induction on r. Indeed, if r=0, then w ia an isofunction and therefore, we have :

$$dw(X_0) = X_0 w$$

This shows that the above formula is true for r=0. We assume that is true for r=1 and using the operators L_x and i_x we obtain the above formula.

5.4 : CONNECTION BETWEEN ISOSYMPLECTIC GEOMETRY AND LIE-ISOTOPIC ALGEBRAS

We now study the **isodifferential calculus of p-isoforms**. Let $\hat{\Phi}_1 = A * \partial x$ be a one-isoform. We define as the **isoexterior derivative of** $\hat{\Phi}_1$ (also called **isoexterior differential**) and denoted with $\partial\hat{\Phi}_1$, the two-isoform

$$\hat{\Phi}_2 = \partial\hat{\Phi}_1 = \frac{\partial(A_{i_1} T^{i_1}{}_{j_1})}{\partial x^{i_2}} T^{i_2}{}_{j_2} \partial x^{j_1} \wedge \partial x^{j_2} = \tag{5.4.1}$$

$$= \left(\frac{\partial A_{i_1}}{\partial x^{i_2}} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} + A_{i_1} \frac{\partial T^{i_1}{}_{j_1}}{\partial x^{i_2}} T^{i_2}{}_{j_2} \right) \partial x^{j_1} \wedge \partial x^{j_2} =$$

$$= \frac{1}{2} \delta^{j_1 j_2}_{k_1 k_2} \left(\frac{\partial A_{i_1}}{\partial x^{i_2}} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} + A_{i_1} \frac{\partial T^{i_1}{}_{j_1}}{\partial x^{i_2}} T^{i_2}{}_{j_2} \right) \partial x^{k_1} \wedge \partial x^{k_2}$$

from which one can see that $\partial\hat{\Phi}_1$ is no longer the curl of the vector field A_{i_1}, but something more general, although admitting the conventional formulation as a particular case for $i=1$.

The **isoexterior derivative of a two-isoform**

$$\hat{\Phi}_2 = A_{i_1 i_2} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} \partial x^{j_1} \wedge \partial x^{j_2} \tag{5.4.2}$$

is given by the three-isoform

$$\hat{\Phi}_3 = \partial\hat{\Phi}_2 = \left(\frac{\partial A_{i_1 i_2}}{\partial x^{i_3}} T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} T^{i_3}{}_{j_3} + A_{i_1 i_2} \frac{\partial T^{i_1}{}_{j_1}}{\partial x^{i_3}} T^{i_2}{}_{j_2} T^{i_3}{}_{j_3} + \right.$$

$$\left. + A_{i_1 i_2} T^{i_1}{}_{j_1} \frac{\partial T^{i_2}{}_{j_2}}{\partial x^{i_3}} T^{i_3}{}_{j_3} \right) \partial x^{j_1} \wedge \partial x^{j_2} \wedge \partial x^{j_3} \qquad (5.4.3)$$

It is easy to see that the isoexterior derivative of the isoexterior product of a p-isoform Φ_p and a q-isoform Φ_q is given by

$$\partial(\hat{\Phi}_p \wedge \hat{\Phi}_q) = (\partial\hat{\Phi}_p) \wedge \hat{\Phi}_q + (-1)^p \hat{\Phi}_p \wedge (\partial\hat{\Phi}_q) \qquad (5.4.4)$$

A p-isoform $\hat{\Phi}_p$ shall ce called **isoexact** when there exists a (p-1) form $\hat{\Phi}_{p-1}$ such that

$$\hat{\Phi}_p = \partial\hat{\Phi}_{p-1} \qquad (5.4.5)$$

Similarly, a p-isoform $\hat{\Phi}_p$ shall be called **isoclosed** when

$$\partial\hat{\Phi}_p = 0 \qquad (5.4.6)$$

The most significant result of this section can be expressed as follows.

LEMMA 5.4.1 ((Isotopic Poincaré Lemma ; Santilli (1988a,b), (1991b). Under sufficient regularity and continuity condi-

tions, the Poincaré Lemma admits an infinite number of isotopic images i.e. given an exact p-form $\Phi_p = d\Phi_{p-1}$, there exists an infinite number of isotopies of Φ_{p-1} into isoforms $\hat{\Phi}_{p-1}$

$$\Phi_{p-1} \Longrightarrow \hat{\Phi}_{p-1} \tag{5.4.7}$$

with consequential isotopies of the p-form

$$\Phi_p = d(\Phi_{p-1}) \Longrightarrow \hat{\Phi}_p = \partial(\hat{\Phi}_{p-1}) \tag{5.4.8}$$

for which the isoexterior derivative of the isoexact p-isoforms are identically null,

$$\partial(\partial\Phi_{p-1}) \equiv 0 \tag{5.4.9}$$

PROOF : Consider an isoexact two-isoform

$$\hat{\Phi}_2 = \partial\hat{\Phi}_1 = \partial(A_i T^i{}_j \partial x^j) \tag{5.4.10}$$

Then, under the necessary regularity and continuity conditions, its isoexterior derivative

$$\partial\hat{\Phi}_2(\partial\hat{\Phi}_1) = \left(\frac{\partial^2 A_{i_1}}{\partial x^{i_2} \partial x^{i_3}} T^{i_1}{}_{j_1} T^{i_3}{}_{j_3} + \frac{\partial A_{i_1}}{\partial x^{i_2}} \frac{\partial T^{i_1}{}_{j_1}}{\partial x^{i_3}} T^{i_2}{}_{j_2} T^{i_3}{}_{j_3} + \right.$$

$$\left. + \frac{\partial A_{i_1}}{\partial x^{i_2}} T^{i_1}{}_{j_1} \frac{\partial T^{i_2}{}_{j_2}}{\partial x^{i_3}} T^{i_3}{}_{j_3} \right) \partial x^{j_1} \wedge \partial x^{j_2} \wedge \partial x^{j_3} \tag{5.4.11}$$

is identically null for all infinitely possible isotopic elements, as the reader can verify via simple but tedious calculations based on the antisymmetrization of all indeces. An iteration of the procedure then proves the lemma at anny (finite) order p. QED.

In short, the existence of consistent isotopies of the Poincaré Lemma proves the consistency of the isotopic generalization of the conventional and exterior calculus under consideration here.

The mathematical relevance of Lemma 5.4.1 is provided by the fact that the abstract, realization-free axioms

$$\Phi_2 = d\Phi_1 \ , \ d\Phi_2 \equiv 0 \qquad \text{(X.X.12a)}$$

$$\Phi_3 = d\Phi_2 \ , \ d\Phi_3 \equiv 0, \text{ etc.} \qquad \text{(X.X.12b)}$$

admit the conventional realization based on an ordinary manifold, as well as an infinite number of additional realizations for each given original form which can be readily identified via our isomanifolds. The latter realizations are generally inequivalent owing to the generally different isotopic elements or isounits.

The conventional Poincaré Lemma constitutes a geometric foundation of Galilei's, Einstein's special and Einstein's general relativities for the exterior problem in vacuum. Also the isotopic Poincaré Lemma constitutes a geometric foundation of the isotopic coverings of the above

relativities for the interior dynamical problem within physical media.

Note that, for each given, conventional realization of axioms (5.4.12), there exist an infinite number of isotopies which are all geometrically equivalent, but physically inequivalent, because they characterize different integro-differential system with inequivalent solutions.

We shall now consider some cases of exact isoclosed isoforms. Consider a one-isoform $\hat{\Phi}_1$ on $T^*M_1(A)$. Then $d\hat{\Phi}_1 \equiv 0$, iff

$$\frac{1}{2}\,\delta^{j_1 j_2}_{k_1 k_2}\left(\frac{\partial A_{i_1}}{\partial x^{i_2}}\, T^{i_1}{}_{j_1} T^{i_2}{}_{j_2} + A_{i_1} \frac{\partial T^{i_1}{}_{j_1}}{\partial x^{i_2}}\, T^{i_2}{}_{j_2} \right) \equiv 0 \quad (5.4.13)$$

namely, the isoclosure of a one-isoform does not imply that the conventional curl of the vector A is null.

Similarly, given a exact two-isoform $\hat{\Phi}_2 = \partial\hat{\Phi}_1$, the property $\partial\hat{\Phi}_2 \equiv 0$ holds iff

$$\delta^{j_1 j_2 j_3}_{k_1 k_2 k_3}\left(\frac{\partial^2 A_{i_1}}{\partial x^{i_2}\partial x^{i_3}}\, T^{i_1}{}_{j_1} T^{i_3}{}_{j_3} + \frac{\partial A_{i_1}}{\partial x^{i_2}}\, \frac{\partial T^{i_1}{}_{j_1}}{\partial x^{i_3}}\, T^{i_2}{}_{j_2} T^{i_3}{}_{j_3} + \right.$$

$$\left. + \frac{\partial A_{i_1}}{\partial x^{i_2}}\, T^{i_1}{}_{j_1} \frac{\partial T^{i_2}{}_{j_2}}{\partial x^{i_3}}\, T^{i_3}{}_{j_3} \right) \equiv 0 \qquad (5.4.14)$$

We are now equipped to identify the desired geometry. Let us review the interplay between exact symplectic two-forms and Lie-isotopic algebras (see Santilli (1982a) for

details). Recall that a conventional two-form on an even, 2n-dimensional manifold $T^*M_2(A)$ with covariant-geometric tensor $\Omega_{i_1i_2}$

$$\Phi_2 = \frac{1}{2}\Omega_{i_1i_2}dx^{i_1}\wedge dx^{i_2} \qquad (5.4.15)$$

characterizes the algebra brackets among functions A(x) and B(x) on $T^*M_2(A)$

$$[A\hat{,}B] = \frac{\partial A}{\partial x^{i_1}}\Omega^{i_1i_2}\frac{\partial B}{\partial x^{i_2}} \qquad (5.4.16)$$

where the contravariant-algebraic tensor $\Omega^{i_1i_2}$ is given by the familiar rule

$$\Omega^{i_1i_2} = \left\{\left|\Omega_{j_1j_2}\right|^{-1}\right\}^{i_1i_2} \qquad (5.4.17)$$

Now, the integrability conditions for two-form (5.4.15) to be an exact symplectic two-form are given by

$$\Omega_{i_1i_2} + \Omega_{i_2i_1} \equiv 0 \qquad (5.4.18a)$$

$$\frac{\partial\Omega_{i_1i_2}}{\partial x^{i_3}} + \frac{\partial\Omega_{i_2i_3}}{\partial x^{i_1}} + \frac{\partial\Omega_{i_3i_1}}{\partial x^{i_2}} \equiv 0 \qquad (5.4.18b)$$

The above conditions are equivalent to the integrability conditions

$$\Omega^{i_1i_2} + \Omega^{i_2i_1} \equiv 0 \qquad (5.4.19a)$$

$$\Omega^{i_1 k} \frac{\partial \Omega^{i_2 i_3}}{\partial x^k} + \Omega^{i_1 k} \frac{\partial \Omega^{i_2 i_3}}{\partial x^k} + \Omega^{i_1 k} \frac{\partial \Omega^{i_2 i_3}}{\partial x^k} \equiv 0 \qquad (5.4.19b)$$

for generalized brackets (5.4.16) to be Lie-isotopic, i.e. verify the Lie algebra axioms in their most general possible, classical, regular realization on $T^*M_2(A)$

$$[A \hat{,} B] + [B \hat{,} A] \equiv 0 \qquad (5.4.20a)$$

$$[[A \hat{,} B] \hat{,} C] + [[B \hat{,} C] \hat{,} A] + [[C \hat{,} A] \hat{,} B] \equiv 0 \qquad (5.4.20b)$$

Thus the exact character of the two form $\Phi_2 = d\Phi_1$ implies its closure $d\Phi_2 \equiv 0$ (Poincaré Lemma), which, in turn, guarantees that the underlying brackets are Lie-isotopic with the canonical case being a trivial particular case (see the analytic, algebraic, and geometric proofs of Santilli (1982),

Lema 5.4.1 establishes that all the above results on the conventional exterior calculus persist under isotopies. Our objective is than that of using the isotopies for the identification of the isounit of the Lie-isotopic algebra directly in the structure of the brackets.

CHAPTER VI

ISOMAPPING BETWEEN ISOMANIFOLDS

6.1 BASIC PROPERTIES OF ISOMAPPINGS

DEFINITION 6.1.1 Let M and N be two differentiable isomanifolds. We consider the mapping Φ of M onto N, that is

$$\Phi : M \longrightarrow N \quad , \quad \Phi : P \longmapsto \Phi(P)$$

which is called **isomapping**. This isomapping Φ is called **differentiable at the point** P, if for every neighborhood U at the point P, there exists a neighborhood $\Phi(U)$ at the point $\Phi(P)$ such that the isofunction

$$g \circ \Phi \in D^0(U) \qquad \forall \; g \in D^0\big(\Phi(U)\big)$$

If the isomapping Φ is differentiable for all the points of the isomanifold M, then Φ is called **differentiable on the whole** M.

REMARK 6.1.1 In the above we have considered differentiable isomapping between two isomanifolds. In a similar manner we

can define differentiable isomapping of class k or briefly C^k. Finally, if the differentiability of Φ is zero or C^0, then the isomapping is continous

REMARK 6.1.2 We assume that M and N are analytic isomanifolds. In a similar way as we have defined a differentiable isomapping, we can define analytic isomapping between M and N.

REMARK 6.1.3 Every analytic isomapping between two analytic isomanifolds M and N is a differentiable isomapping and the inverse is not always true, that is a differentiable isomapping between two analytic isomanifolds M and N is not always analytic.

6.1.1 Expression of an isomapping in local isocordinate systems. Let Φ be an isomapping between two differentiable isomanifolds M and N, that is

$$\Phi : M \longrightarrow N \quad , \quad \Phi : P \longmapsto \Phi(P)$$

Let U be a neighborhood of the point P with $(x^1,\dots,x^m)$ local isocoordinate system and U′ a neighborhood of $\Phi(P)$ such that $U' \subseteq \Phi(P)$ with local isocoordinate system. Therefore we have the relations

$$y^i = \varphi^i(x^1,\dots,x^m) \quad i=1,\dots,n \qquad (6.1.1)$$

The relations (6.1.1) give the expression of Φ in local isocoordinate systems. The isomapping Φ is differentiable of order k at the point P if there are all the partial derivatives of (6.1.1) until of order k at the point P.

DEFINITION 6.I.2 Let **M** and **N** be two differentiable isomanifolds and Φ a diferentiable isomapping such that Φ is a homeomorphism. Then Φ is called **differentiable isohomeomorphism.**

REMARK 6.1.4 In the above we have assumed that Φ is a differentiable isomapping of order infinite or C^{∞}. If we assume that Φ is a homeomorphism of differentiability of order k, or C^{k}, then it is called **isohomeomorphism of differentiability of order k**.

REMARK 6.I.5 If the isomanifold **M** and **N** are analytic and the isomapping $\Phi : M \longrightarrow N$ is analytic, then Φ is called **isoanalytic**.

DEFINITION 6.I.3 Let **M** be a differentiable isomanifold. A defferentiable isohomeomorphism of **M** onto **M** is called **differentiable isotransformation** or simply **isotransformation**.

DEFINITION 6.1.4 Let **M** be a differentiable isomanifold. We consider a differentiable isomapping α of the open isointerval $I \subseteq \hat{R}$ into **M**, that is

$$\alpha : I \longrightarrow M \quad , \quad \alpha : t \in I \longmapsto \alpha(t) \in M$$

This isomapping is called **isocurve** on **M**.

REMARK 6.1.6 If the isomapping α is of class C^{k}, then α is called **isocurve** of class C^{k}. In some cases we consider, for the definition of the isocurve, that the isointerval I is

closed, that is

$$I = [a,b] \leq \hat{R} \quad a,b\in\hat{R}$$

This is true under the condition that the isomapping α can be extended to an open isointerval $I_1 \supseteq I$ of $\hat{R}$.

DEFINITION 6.1.5 Let o(t) be an isocurve on the differentiable isomanifold M. Let P be a point of the isomanifold and f a differentiable isofunction defined on a neighborhood of P. We assume that there exist the limit

$$[o'(t)](f) = \lim_{h \to 0} \frac{1}{h}\left[f(\alpha(t+h))-f(o(t)) \right]$$

where t is the value of the parameter which corresponds to the point P.

Let (U,ϕ) be an isochart of M such that $o(t)\in U$ with isocoordinates $(u^1,\dots,u^n)$. Then the isofunction f for the point $q\in U$ has the form

$$f(q) = f\left(u^1(q),\dots,u^N(q)\right)$$

Therefore f can be described as an isofunction on the open subset $\phi(U)\subseteq\hat{R}^n$. The part of the isocurve o(t) on U can be expressed as follows

$$u^i\left(o(t)\right) = u^i(t) \quad i=1,\dots,n$$

From the known derivation's ruler we obtain

$$[o'(t)](f) = \sum_{i=1}^{n} \frac{\partial f}{\partial u^i} \left.\frac{du^i}{dt}\right|_{u^i=u^i(t)} \qquad (6.1.2)$$

$\alpha'(t)$ is called **tangent isovector** of the isocurve at the point $\alpha(t)$.

DEFINITION 6.1.6 The relation (6.1.2) is called **derivative** of the isofunction f along the isocurve α at the point $\alpha(t)$.

DEFINITION 6.1.7 Let P be a point of the differentiable isomanifold M. We consider all the differentiable isocurves $\alpha(t)$ which are defined by

$$\alpha : I \longrightarrow M \quad , \quad \alpha : t \longmapsto \alpha(t) \in M$$

$$0 \in I \text{ and } \alpha(0) = P$$

The set of all these curves is denoted by S on which we define a binary relation " ~ " as follows

$$\alpha \sim \beta \text{ if } \alpha(0) = \beta(0) \text{ and}$$

$$\frac{\overset{e}{d}x^i(\alpha(t))}{dt}\Big|_{t=0} = \frac{dx^i(\beta(t))}{dt}\Big|_{t=0}$$

where $(x^1, \ldots, x^n)$ are the local coordinates of the chart (U, φ) and $0 \in U$.

It can be easily proved that "~" ia an equivalence relation on S, which defines classes. Each class of equivalence is obtained as a tangent isovector of M at the point $P \in M$. Therefore all the classes of equivalence define the **tangent isospace** $T_p(M)$ of M at the point P.

REMARK 6.1.7 Very often we identify the tangent isovector v at the point $P \in M$ with the isocurve $\alpha(t)$ under the condition

that $\alpha'(t)$ is a linear form on the isomodule.

We consider the isocurves α_i, $i=1,\ldots,n$ defined by :

$$\alpha_i = \left[\alpha^i(t) = \alpha^i(P)+t \quad , \quad \alpha^j(t)=\alpha^j(P) \quad , \quad i\neq j \right]$$

It can be easily proved that $\alpha_i'(0)$ is a linear isoform on $D^0(U)$, which can be identified with

$$\left.\frac{\partial}{\partial x_i}\right|_P$$

From the above identification we conclude that the isocurves

$$\alpha_1(t),\ldots,\alpha_n(t)$$

can be considered as a base of the tangent isospace $T_p(M)$.

DEFINITION 6.1.8 Let Φ be a differentiable isomapping between two differentiable isomanifolds M and N. Let (U,φ) be an isochart of M and $P\in U$. From Φ we obtain the point $\Phi(P)\in N$ and consider the open neighborhood U' of N such that $\Phi(P)\in U'\subset \Phi(U)$. Let $D^0(U)$ and $D^0(U')$ be the isoalgebras of the differentiable isofunctions on U and U' respectively.

If A is an isovector, then A can be considered as an isooperator on $D^0(U)$, that means

$$A : D^0(U) \longrightarrow D^0(U) \quad , \quad A : g \longmapsto A(g)$$

To the isovector $A\in T_p(M)$ we correspond to the isovector $B\in T_{\Phi(P)}(N)$ defined as an isooperator on $D^0(U')$ as follows

$$B : D^0(U') \longrightarrow D^0(U') \quad , \quad B : h \longmapsto B(h)$$

The B(h) is defined by the relation

$$B(h)_{\Phi(P)} = A(h\circ\Phi)_P$$

Now we have constructed an isomapping, denoted by $(\Phi_*)_P$, of $T_P(M)$ into $T_{\Phi(P)}(N)$ defined as follows

$$(\Phi_*)_P : T_P(M) \longrightarrow T_{\Phi(P)}(N)$$

$$(\Phi_*)_P : A \longmapsto (\Phi_*)_P(A)=A$$

This isomapping $(\Phi_*)_P$ is called derivative of Φ at the point P.

THEOREM 6.1.1 The derivative isomapping $(\Phi_*)_P$ of the isomapping Φ between the isomanifolds M and N at the point $P\in M$ is an isolinear mapping of $T_P(M)$ into $T_{\Phi(P)}(N)$.

<u>Proof</u> Let o be an isocurve on the isomanifold M, that is :

$$o : \hat{I} \longrightarrow M \quad , \quad \Phi : \hat{0}\in\hat{I} \longrightarrow \Phi(\hat{0})=P$$

where $\hat{I}$ is an open isointerval of M. From the isomapping

$$\Phi : M \longrightarrow N$$

we obtain the isocurve $\beta=\Phi\circ o$ on N and therefore $\beta(0)=\Phi(P)$. Let A be the tangent isovector of o at the point P. By the meaning of the isomapping derivative $(\Phi_*)_P$ we obtain the tangent isovector B of β at the point $\Phi(P)$. From the constraction of $(\Phi_*)_P$ we have

$$(\Phi_*)_P : T_P(M) \longrightarrow T_{\Phi(P)}(N)$$

$$(\Phi_*)_P : \hat{\lambda}_1 A_1 + \hat{\lambda}_2 A_2 \longmapsto (\Phi_*)_P(\hat{\lambda}_1 A_1 + \hat{\lambda}_2 A_2) =$$
$$= \hat{\lambda}_1 (\Phi_*)_P(A_1) + \hat{\lambda}_2 (\Phi_*)_P(A_2)$$

where $\hat{\lambda}_1, \hat{\lambda}_2 \in \hat{\mathbb{R}}$ and $A_1, A_2 \in T_P(M)$.

From the above we conclude that $(\Phi_*)_P$ is linear isomapping between the tangent isospaces.

PROBLEM 6.1.1 Let $(\Phi_*)_P$ be the isomapping derivative between the tangent isospaces $T_P(M)$ and $T_{\Phi(P)}(N)$ of the isomanifolds M and N at the points P anf $\Phi(P)$ respectively. Determine $(\Phi_*)_P$ by a matrix.

<u>Solution</u> Let (U,Ψ) and (U',Ψ) be two isocharts of M and N respectively such that $P \in U$ and $\Phi(P) \in U'$ and $\Phi(U) \subset U'$. Let $(x^1, \dots, x^n)$ and $(y^1, \dots, y^n)$ be the isocoordinates on U and U' respectively. For the isohomeomorphisms Ψ and Ψ' we have

$$\Psi : U \longrightarrow \Psi(U) \subseteq \hat{\mathbb{R}}^n \quad , \quad \Psi : q \longmapsto \Psi(q) =$$
$$= \left(x^1(q), \dots, x^n(q)\right) = (x^1, \dots, x^n)$$

$$\Psi' : U' \longrightarrow \Psi'(U') \subseteq \hat{\mathbb{R}}^m \quad , \quad \Psi' : r \longmapsto \Psi'(r) =$$
$$= \left(y^1(r), \dots, y^m(r)\right) = (y^1, \dots, y^m)$$

Hence the isomapping Φ locally can be expressed as follows :

$$y^k = y^k(x^1, \dots, x^n) \qquad k=1, \dots, m$$

It is known that the isovectors

$$e_i = \left(\frac{\partial}{\partial x^i} \right)_P \qquad i=1,\dots,n$$

form a base for the tangent isospace $T_P(M)$.
Similarly the isovectors :

$$E_j = \left(\frac{\partial}{\partial y^j} \right)_{\phi(P)} \qquad j=1,\dots,m$$

form a base for the tangent isospace $T_{\phi(P)}(N)$. Therefore the derivetive isomapping $(F^*)_P$ for these isovectors implies

$$E_j = \frac{\partial}{\partial y^j} = \frac{\partial \varphi^j}{\partial x^1} \frac{\partial}{\partial x^1} + \dots + \frac{\partial \varphi^j}{\partial x^n} \frac{\partial}{\partial x^n} \qquad j=i,\dots,m$$

from which we have :

$$T = \begin{pmatrix} \frac{\partial \varphi^1}{\partial x^1} & \frac{\partial \varphi^1}{\partial x^2}, & \dots, & \frac{\partial \varphi^1}{\partial x^n} \\ \frac{\partial \varphi^2}{\partial x^1} & \frac{\partial \varphi^2}{\partial x^2}, & \dots, & \frac{\partial \varphi^2}{\partial x^n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial \varphi^m}{\partial x^1} & \frac{\partial \varphi^m}{\partial x^2}, & \dots, & \frac{\partial \varphi^m}{\partial x^n} \end{pmatrix}_P$$

Hence $(\Phi_*)_P$ is represented by the matrix

$$S = {}^tT = \begin{pmatrix} \dfrac{\partial\varphi^1}{\partial x^1} & \dfrac{\partial\varphi^2}{\partial x^1}, & \ldots, & \dfrac{\partial\varphi^m}{\partial x^1} \\ \dfrac{\partial\varphi^1}{\partial x^2} & \dfrac{\partial\varphi^2}{\partial x^2}, & \ldots, & \dfrac{\partial\varphi^m}{\partial x^2} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial\varphi^1}{\partial x^n} & \dfrac{\partial\varphi^2}{\partial x^n}, & \ldots, & \dfrac{\partial\varphi^m}{\partial x^n} \end{pmatrix}_P \qquad (6.1.3)$$

REMARK 6.1.8 If $N=\hat{\mathbb{R}}$, the $T_{\Phi(P)}(N)$ can be identified with $\hat{\mathbb{R}}$ and in this case $(\Phi_*)_P$ is an isolinear form on $T_P(M)$. In this case $\Phi_*=\partial\Phi$ can be considered as defferentiable one-isoform on the isomanifold.

If $A\in T_P(M)$, then the tangent isovector $(\Phi_*)_P(A)$ and the tangent isovector B defined by the relation

$$B : f \longrightarrow \frac{d}{dt}\, f(\Phi(P)+tA(\Phi))$$

where $f\in D^0(\hat{\mathbb{R}})$, determine the same number $f'(\Phi(P))A(\Phi)$.

REMARK 6.1.9 If the differentiable isomapping Φ between the two differentiable isomanifolds M and N at the point $P\in M$ has the property such that the derivative isomapping $(\Phi_*)_P$ of $T_P(M)$ into $T_\Phi(N)$ is one-to-one, then Φ is called **regular** at the point P. In this case the matrix (6.1.3) has rank equal to the minimum between the two numbers m,n.

REMARK 6.1.10 We consider the isocurve α on the differentiable isomanifold M, that is

$$\alpha : I \longrightarrow M \ , \ \alpha : t \longmapsto \alpha(t) \ , \ \alpha : t_0 \longmapsto \alpha(t_0)=P$$

which is canonical at the point $P\in\alpha$. In order to be canonical at the point P it is necessary at least one of the derivatives of the functions

$$u^i(\alpha(t)) = u^i(t)$$

where $(u^1,\ldots,u^n)$ the local coordinate system on the chart (U,φ) where $U\subset M$ such that $\alpha(t)\subset U$, must be different than zero at P.

THEOREM 6.1.2 Let Φ be an isohomeomorphism between the two isomanifolds M and N. We assume that the isomapping derivative $(\Phi_*)_P$ is an isoisomorphism of T_PM onto $T_{\Phi(P)}N$, then there are neighborhoods U and U′ of the points P and $\Phi(P)$ respectively such that Φ/U is a differentiable isohomeomorphism of U onto U′.

<u>Proof.</u> The restriction of the isohomeomorphism, that is Φ/U can be expressed by the relations

$$y^i = \varphi^i(x^1,\ldots,x^n) \quad i=1,\ldots,n$$

since the isomapping derivative $(\Phi_*)_P$ is an isoisomorphism of $T_P(M)$ onto $T_{\Phi(P)}(N)$, it implies that the Jacobian

$$\frac{D(\varphi^1,\dots,\varphi^n)}{D(x^1,\dots,x^n)} = \begin{vmatrix} \frac{\partial\varphi^1}{\partial x^1} & ,\dots, & \frac{\partial\varphi^1}{\partial x^n} \\ \dots & \dots & \dots \\ \frac{\partial\varphi^n}{\partial x^1} & ,\dots, & \frac{\partial\varphi^n}{\partial x^n} \end{vmatrix} \neq 0$$

The theorem of the inverse isomapping implies the proof.

6.2 CONNECTION BETWEEN ISOTENSOR FIELDS AND ISOMAPPING

DEFINITION 6.2.1 Let Φ be a differentiable isomapping between the two isomanifolds M and N. We consider two isovector fields X and Y of M and N respectively. If for every $P \in M$ we have

$$(\Phi_*)_P(X_P) = Y_{\Phi(P)}$$

then the two isovector field X and Y are called **connected** trough the isomapping Φ and this is denoted by

$$(\Phi_*)_P = Y \qquad (6.2.1)$$

PROPOSITION 6.2.I Let X_i, Y_i, i=1,2 be two connected isovector fields of the differentiable isomanifolds M and N respectively with respect to the differentiable isomapping Φ

between M and N. The following relation

$$\Phi_*([X_1,X_2]) = [Y_1,Y_2] \qquad (6.2.2)$$

holds.

Proof. It is known the following

$$(\Phi_*)_P((X_1)_P) = (Y_1)_{\Phi(P)} \quad , \quad (\Phi_*)_P((X_2)_P) = (Y_2)_{\Phi(P)}$$

From the definition of the Lie bracket, the properties of the derivative isomapping and the (6.2.3) we conclude that

$$(\Phi_*)_P((X_1)_P,(X_2)_P) = [Y_1,Y_2] \quad \forall P\in M$$

which implies the relation (6.2.2)

PROPOSITION 6.2.3 Let Φ be a differentiable isotransformation of M. We set $f^*=f\circ\Phi^{-1}$, where $f\in D^0(M)$. Prove that

$$\Phi_*(f(X))=f^*(\Phi_*(X)) \text{ and } f^*(Xf)=\Phi_*(X)(f^*) \ , \ X\in D^1(M) \quad (6.2.4)$$

Proof. If X and Y are two isovector fields, which are connected by the isotransformation Φ, By this property of Φ we have

$$(Yf)\circ\Phi = X(f\circ\Phi) \qquad (6.2.5)$$

where $f\in D^0(M)$. If we have under the consideration (6.2.5) and some properties of the derivative isomapping then we have proved (6.2.4).

DEFINITION 6.2.2 Let $\Phi : M \longrightarrow N$ be an isomapping between the two differentiable isomanifolds M and N. Let ω be an isoexterior k-form of N, Then there exists an isoexterior k-form of M, denoted by $\Phi^*(\omega)$, such that

$$\omega(Y_1,\ldots,Y_k) = \Phi^*(\omega)(X_1,\ldots,X_r)\circ\Phi \qquad (6.2.6)$$

where $Y_1,\ldots,Y_r \in D^1(N)$ are connected of $X_1,\ldots,X_r \in D^1(M)$ respectevely, through the isomapping Φ, that means

$$\Phi_*(X_i) = Y_i \quad i=1,\ldots,r \qquad (6.2.7)$$

The relation (6.2.6), for an abirtrary point $P\in M$, can be expressed as follows

$$\Phi^*(\omega)\{A_1,\ldots,A_r\} = \omega\ \{(\Phi_*)_p(A_1),\ldots,(\Phi_*)_p(A_r)\} \qquad (6.2.8)$$

where $A_i \in T_p(M)$, $i=1,\ldots,r$ are isovectors of the isotangent space $T_p(M)$.

REMARK 6.2.1 Let $f\in D^0(N)$ be an isofunction of N. We set $\Phi^* f = f\circ\Phi$. If $\omega\in\Lambda(N)$, where $\Lambda(N)$ is the isoexterior algebra of N, then

$$\Phi^*(\omega)\in\Lambda(M)$$

It is known that

$$\Lambda(M) = \underset{r=1}{\oplus}\Lambda(M) \quad , \quad \Lambda(N) = \underset{k=1}{\oplus}\Lambda(N)$$

where $\Lambda^0(M) = D^0(M)$, $\Lambda(N) = D^0(N)$.

PROPOSITION 6.2.2 If $\omega_1,\omega_2\in\Lambda(N)$, then the following relations hold

$$\Phi^*(\omega_1\wedge\omega_2) = \Phi^*(v_1)\wedge\Phi^*(\omega_2) \quad , \quad d(\Phi^*\omega) = \Phi^*(d\omega)$$

where $\omega\in\Lambda(N)$ and Φ is an isomapping between the two differentiable isomanifolds M and N.

<u>Proof</u> This is an immediate consequence of the definitions of the exterior product of two isoexterior forms, of the

derivation of an isoexterior form and of the derivative isomapping.

REMARK 6.2.2 If $T \in sD(N)$, that means T is a covariant isotensor field of N, then we can define with the semilar manner the $\Phi^*(T) \in sD(M)$.

If $\Omega \in aD(M)$, that is Ω is a contravariant isotensor field of M, then, by means of the isomapping between M and N, it can be determined a contravarian isotensor field of N denoted by $\Phi_*(\Omega)$. Therefore we have

$$\Phi_*(\Omega) \in oD(N)$$

DEFINITION 6.2.3 Let Φ be a differentiable isohomeomorphism of M, that is M=N. If we have

$$\Phi^*(t) = T \in sD(M)$$

then T is called **invariant** by Φ. Similarly if we have

$$\Phi_*(\Omega) = \Omega \in oD(N)$$

then Ω is called **invariant** by Φ.

PROBLEM 6.2.1 Let Φ be differentiable isomapping Φ between the two isomanifolds M and N. If ω is an isoexterior r-form of N whose the corrsponding through Φ is $\Phi^*\omega$ of M. Determine $\Phi^*\omega$ in local coordinate system.

<u>Solution.</u> Let P be a point of M whose the corresponding of N is $\Phi(P)$. Let U and U′ be two neighborhoods of P and $\Phi(P)$ respectively such that $\Phi(U) \subset U'$. The isomapping Φ can be

expressed by local isocoordinates as follows

$$y^k = \varphi^k(x^1,\ldots,x^m) \qquad k=1,\ldots,n \tag{6.2.9}$$

where $(x^1,\ldots,x^m)$ local isocoordinates around P in U and $(y^1,\ldots,y^n)$ local isocoordinates around $\Phi(P)$ in U'.
The restriction of the isoexterior r-form on U' can be written :

$$\omega/_{U'} = \Sigma\, g_{i_1\ldots i_r}\, dy^{i1}\wedge\ldots\wedge dy^{ir} \quad 1\le i_1<\ldots<i_r\le n \tag{6.2.10}$$

where $g_{i_1\ldots i_r} \in D^0(U')$

The restriction of the isoexterior r-form $\Phi^*\omega$ on U takes the form

$$\Phi^*\omega/_U = \Sigma\, f_{h_1\ldots h_r}\, dx_1^{h1}\wedge\ldots\wedge dx^{hr} \quad 1\le h_1<\ldots<h_r\le n \tag{6.2.11}$$

The relation (6.2.11) can be obtained by (6.2.10) by means of

$$y^k=\varphi^k(x^1,\ldots,x^m)\ ,\ dy^k = \sum_{i=1}^{m} \frac{\partial\varphi^k}{\partial x^i}\, dx^i \quad k=1,\ldots,n$$

6.3. SUBISOMANIFOLDS

DEFINITION 6.3.1 Let M and N be two differentiable isomanifolds. If $M\subseteq N$, M and N considered as sets and the induced isomapping

$$i : M \longrightarrow N\ ,\quad i : P \longrightarrow i(P)=P$$

is regular for every point $p \in M$, then the isomanifold M is called **subisomanifold** of N.

EXAMPLE 6.3.2 Let $\hat{S}^5$ be the isosphere defined by the equation

$$\hat{S}^5 = \left[(x^1,\ldots,x^6) \subset \hat{\mathbb{R}}^6 \ / \ x_1^2 + \ldots + x_6^2 = 1 \right]$$

Prove that $\hat{S}^5$ is a subisomanifold of the Euclidean isospace $\hat{\mathbb{R}}^6$.

Solution. The hypersphere $\hat{S}^5$ is a subisomanifold of $\hat{\mathbb{R}}^6$ and at the same time topological subspace of $\hat{\mathbb{R}}^6$.

REMARK 6.3.1 Let M be a subisomanifold of N. It is possible M showed not be a topological subspace of N.

THEOREM 6.3.4 Let M be a subisomanifold of N. If $P \in N$, then there exist a local isocoordinate system $(x^1,\ldots,x^n)$ around P in a neighborhood U of N such that

$$x^1(P) = \ldots = x^n(P) = 0$$

and the same time the set

$$U' = \left[q \in U \ / \ x^j(q) = 0 \ \ m+1 \leq j \leq n \right]$$

with the restriction of $(x^1,\ldots,x^n)$ on U' to form a local isocoordinate system around $P \in M$ in $U' \subset M$.

Proof Let U' and U be two neighborhoods of the point P considered as a point of the isomanifolds M and N respectively. Let $(y^1,\ldots,y^m)$ and $(z^1,\ldots,z^n)$ be local isocoordinate systems on U' and U respectively with the

properties

$$y^i(P) = 0 \ , \ i=1,\dots,m \ , \ z^j(P)=0 \ , \ j=1,\dots,n$$

The induced isomapping

$$i : M \longrightarrow N \quad , \quad i : P \longmapsto i(P) \quad \forall P \in M$$

in the neighborhood of P can be expressed by meaning of the relations

$$z^j = \varphi^j(y^1,\dots,y^m) \qquad j=1,\dots,n$$

The Jacobian matrix

$$J = \begin{pmatrix} \dfrac{\partial\varphi^1}{\partial y^1} & \cdots\cdots & \dfrac{\partial\varphi^1}{\partial y^m} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial\varphi^n}{\partial y^1} & \cdots\cdots & \dfrac{\partial\varphi^n}{\partial y^m} \end{pmatrix}$$

of the system has rank m at the point P, since i is regular at this point.

Without loss of generality we can assume that the square matrix

$$\begin{pmatrix} \dfrac{\partial\varphi^1}{\partial y^1} & \cdots\cdots & \dfrac{\partial\varphi^1}{\partial y^m} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial\varphi^m}{\partial y^1} & \cdots\cdots & \dfrac{\partial\varphi^m}{\partial y^m} \end{pmatrix}$$

has determinant different than zero at the point. Therefore at the neighborhood (0,...,0) we obtain

$$y^i = f^i(z^1,\dots,z^n) \qquad 1 \le i \le m$$

Each of the function f^i is differentiable. If we set

$$x^i = z^i \qquad i=1,\dots,m$$

$$x^j = z^j - \varphi^j(f^1(z^1,\dots,z^m),\dots,f^m(z^1,\dots,z^m))$$

then it can be easily proved that

$$D\left[\left(\partial x^i/\partial y^l\right)\right] \ne 0 \quad i=1,\dots,m \ , \quad l=1,\dots,m$$

$$D\left[\left(\partial x^j/\partial z^k\right)\right] \ne 0 \quad j=1,\dots,n \ , \quad k=1,\dots,n$$

where D is the determinant of the matrices

$$\left(\partial x^i/\partial y^l\right) \qquad i=1,\dots,m \ , \quad l=1,\dots,m$$

$$\left(\partial x^j/\partial z^k\right) \qquad j=1,\dots,n \ , \quad k=1,\dots,n$$

Therefore $(x^1,\dots,x^n)$ is the required local isocoordinate system q.e.d.

DEFINITION 6.3.2 Let M and N be two differentiable isomanifolds. Let Φ be a differentiable isomapping of M into N, that is

$$\Phi : M \longrightarrow N$$

If the derivative isomapping $(\Phi_*)_p$, $\forall P\in M$, is one-to-one, then Φ is called **isoimersion** and M is called **isoimersed isomanifold** in the isomanifold N.

If at the same time is one-to-one then Φ is called **isoembeding** and M is called **isoembended isomanifold** in the isomanifold N.

REMARK 6.3.2 Let Φ be an isoimersion of the isomanifold into the isomanifold N. Since $(\Phi_*)_p$ is one-to-one for every $P\in M$, we conclude that

$$\dim(T_pM) \le \dim(T_{\Phi(P)}N)$$

and therefor $\dim M \le \dim N$.

There exists a neighborhood U of the point P with local isocoordinate system, $(x^1,\ldots,x^m)$, and a neighborhood U of the point $\Phi(P)$ with local isocoordinate system $(y^1,\ldots,y^n)$ such that we have

$$y^i(\Phi(Q)) = x^i(Q) \quad \forall Q\in U \quad i=1,\ldots,m$$

Specially it is a homeomorphism of U onto U'. The isomerced isomanifold M can be considered as subisomanifold.

REMARK 6.3.3 The isomanifold M is a Hausdorff space and therefore has a topology denoted by T_1. On the other hand the isomapping $\Phi : M \longrightarrow N$ induces on M another topology denoted by T_2. There is an open problem to compare these two topologies T_1 and T_2 on M.

DEFINITION 6.3.3 Let Φ be an isomersion of M into N, that is

DEFINITION 6.4.2 Let S be an isodistribution of a differentiable isomanifold M. If $\forall P \in M$ S_P is a subspace of the isotangent space $T_P(M)$ of dimension $r<n$, where $n=\dim M$, then r is called **dimension** of the isodistribution S.

EXAMBLE 6.4.1 Let $\hat{\mathbb{R}}^4$ be the isoeuclidean manifold. We consider the isovector fields

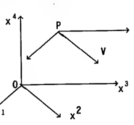
$$X_1 = e^{x_1}\frac{\partial}{\partial x_1}\ ,\ X_2 = e^{x_2}\frac{\partial}{\partial x_2}\ ,\ X_3 = e^{x_3}\frac{\partial}{\partial x_3}$$

Prove that these form an isodistribution of $\hat{\mathbb{R}}^4$. Determine its dimension. Is this an involutive ?

<u>Solution</u> If $P(p_1,p_2,p_3,p_4)\in\hat{\mathbb{R}}^4$, then three isovector fields X_1,X_2,X_3 define the following isovectors of $T_P(\hat{\mathbb{R}}^4)$

$$(X_1)_P=e^{p_1}\left(\frac{\partial}{\partial x_1}\right)_P\ ,\ (X_2)_P=e^{p_2}\left(\frac{\partial}{\partial x_2}\right)_P\ ,\ (X_3)_P=e^{p_3}\left(\frac{\partial}{\partial x_3}\right)_P$$

where (x_1,x_2,x_3) the isoaffine coordinate system on $\hat{\mathbb{R}}^3$. These three isovectors define a subvectorspace V of $T_P(\hat{\mathbb{R}}^4)$ of dimension 3. If we consider V as a subisomanifold of $\hat{\mathbb{R}}^4$, then it represent an isohypersurface of $\hat{\mathbb{R}}^4$.



which is parallel to the coordinate isohyperplan $Ox_1x_2x_3$. Therefore the three isovector fields X_1, X_2, X_3 of $\hat{R}^4$ form an isodistribution, whose dimension is three, that is dimS=3. In order to prove that S is involutive it is necessary that the Lie brackets $[X_1,X_2]$, $[X_2,X_3]$, $[X_1.X_3]$ belong to this isodistribution. These Lie brackets have componets different than zero only for the isovector fields $\partial/\partial x_1$, $\partial/\partial x_2$, $\partial/\partial x_3$. Therefore the distribution S is involutive.

DEFINITION 6.4.2 Let N be a subisomanifold of the differentiable isomanifold M, that is we have the isomapping $\Phi : N \longrightarrow M$. We consider the isodistribution S of M such that to be

$$(\Phi_*)_P(T_P(N)) = S_{\Phi(P)} \quad \forall P\in N$$

The subisomanifold N is called **integrable isomanifold** of S. If there not exist any other integrable isomanifold of S, which contains N, then N is called **maximal integrable isomanifold** of S.

PROPOSITION 6.4.1 Let S be an involutive isodistribution of M. Then for each point P∈M, there exists a unique maximal integrable isomanifold N(P) of S. Every integrable isomanifold through the point P is a subisomanifold of N.

Proof This is an immediate sequence of the definitions of involutive isodistribution, integrable isomanifold and maximal integrable isomanifold associated to an involutive isodistribution.

Similarly we obtain the proposition :

PROPOSITION 6.4.2 Let S be an ivolutive isodistribution of the differentiable isomanifold. Let N be a subisomanifold of whose the connected parts are all integral subisomanifolds of S.

We consider the differentiable isomapping of the isomanifold into the isomanifold M such that

$$f(K) \subset N$$

If N satisfies the second axiom of countability, then f is a differentiable isomapping of K into N.

DEFINITION 6.4.3 Let X be an isovector field of the differentiable isomanifold M. Let u(t) be an isocurve of M, that is

$$u : \hat{I} \longrightarrow M \quad , \quad u : t \longmapsto u(t) \in M$$

u(t) is called **integrable** of the isovector field if for every t_0 of t, the isovector $X_{u(t_0)}$ is isotangent of the isocurve at the point $u(t_0)$.

THEOREM 6.4.1 Let X be an isovector field of the differentiable isomanifold M. For every point $P \in M$, there exists a unique integral isocurve u(t) of X, defined for $|t| < \varepsilon$, where $\varepsilon > 0$ and such that $u(0) = P$.

Proof We obtain a neighborhood U of the point P with local isocoordinate system $(x^1, \ldots, x^n)$. In this case the isovector field X can be written :

$$X = \lambda^1 \frac{\partial}{\partial x^1} + \dots + \lambda^n \frac{\partial}{\partial x^n}$$

where $\lambda^1=\lambda^1(x^1,\dots,x^n),\dots,\lambda^n=\lambda^n(x^1,\dots,x^n)$ isofunctions of U. Every integral isocurve $u(t)=\{x^1(t),\dots,x^n(t)\}$ is a solution of the system :

$$\frac{dx^i}{dt} = \lambda^i(x^1,\dots,x^n) \quad i=1,\dots,n \qquad (6.4.1)$$

According to existance theorem of the solution of the system (6.4.1), there is a unique solution which is satisfied by P for t=0. The solution is valid for $|t|<\varepsilon$. The solution gives the unique integrable isocurve of X having the required properties.

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ABOUT THE BOOK

This is the first mathematical book written on an axiom-preserving generalization of Lie's theory proposed in 1978 by **R. M. Santilli** while at the Department of Mathematics of Harvard University, developed by numerous scholars, and constructed for the characterization of systems which are nonlinear and nonlocal in coordinates, velocities and other quantities, as well as non-Lagrangian and non-Hamiltonian. As such, the Lie-Santilli theory is seeing an increasing number of advanced applications in nuclear, particle and statistical physics, astrophysics, superconductivity, computer modeling, theoretical biophysics and other fields.

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